UNIVERSITÀ
DEGLI STUDI
DI TORINO

ALMA UNIVERSITAS
TAURINENSIS

Founded in 1404

DEPARTMENT OF
ECONOMICS AND STATISTICS
WORKING PAPER SERIES

Quaderni del Dipartimento di Scienze
Economico-Sociali e Matematico-Statistiche

ISSN 2279-7114

THE ZENGA INDEX REVEALS MORE THAN THE GINI AND THE BONFERRONI INDEXES

AN ANALYSIS OF DISTRIBUTIONAL CHANGES AND SOCIAL WELFARE LEVELS

MARIA GIOVANNA MONTI

SIMONE PELLEGRINO

ACHILLE VERNIZZI

Working paper No. 84 - January 2024

THE ZENGA INDEX REVEALS MORE THAN THE GINI AND THE BONFERRONI INDEXES

AN ANALYSIS OF DISTRIBUTIONAL CHANGES AND SOCIAL WELFARE LEVELS

MARIA GIOVANNA MONTI[♣]
SIMONE PELLEGRINO[♦]
ACHILLE VERNIZZI[♥]

9-1-2024

ABSTRACT

In this paper we discuss the characteristics of the Zenga index and describe how it interprets both income inequality and social welfare. The Zenga index is quite different from those commonly used today. The difference is mainly due to the role played by income weights: they depend not only on income ranks but also on income distribution. To achieve our goals, we compare the Zenga index behaviour with that of other two indexes: the Gini and the Bonferroni index. The first issue we address concerns the effect of distributional changes in terms of inequality indexes. This analysis highlights how the Zenga index interprets income inequality differently with respect to the other two indexes. The second issue we consider is to determine the social welfare function associated to the Zenga index. We derive it by employing the method proposed by Dagum in 1990. By employing the Zenga social welfare function, we show that one can go beyond Dagum's remarks: when the income increase takes place in the top part of the income distribution, the social welfare can decrease when mean income increases. We can thus conclude that the Zenga inequality index reveals more than the Gini and Bonferroni indexes; in particular, it is more sensitive for evaluating income changes taking place in different parts of the income distribution.

1. Introduction

In 2007 Zenga presented a new inequality index that is quite different from those commonly used today. The difference is mainly due to the role played by income weights: as we show in this paper, they depend not only on income ranks but also on income distribution.

Given this peculiarity, our main goal is to discuss the characteristics of the Zenga index and to describe how it interprets both income inequality and social welfare. Moreover, the peculiarities of this index emerging from our analysis suggest evaluating its link with deprivation. To achieve our goals, we compare the Zenga index behaviour with that of other two indexes: the Gini and the Bonferroni index. We chose the first one because it is the most widely used inequality index and the second one because it is linked to the Zenga index. After discussing the theoretical peculiarities of these indexes, the empirical sections propose simulations able to underline further details of our theoretical strategy.

[♣] University of Milan, 20122 Milan, Italy.
[♦] Corresponding author. Department of Economics, Social Studies, Applied Mathematics and Statistics – ESOMAS, University of Turin, Corso Unione Sovietica 218bis, 10134 Turin, Italy. Email address: simone.pellegrino@unito.it.
[♥] Department of Economics, Management and Quantitative Methods, University of Milan, Via Conservatorio 7, 20122 Milan, Italy.

The first issue we address concerns the effect of distributional changes in terms of inequality indexes. This analysis highlights how the Zenga index interprets income inequality differently with respect to the other two indexes. We also examine the link between the Zenga index and some conditions that an inequality index has to satisfy. In particular, we consider the Transfer Sensitivity and the Principle of Positional Transfer Sensitivity, two principles conceptually linked to the Pigou-Dalton principle. After recalling that the Bonferroni index satisfies the Positional Transfer Sensitivity Principle whilst the Gini index does not, we show that this Principle is also satisfied by the Zenga index. This is our first result.

The second issue we consider is to determine the social welfare function associated to the Zenga index. We derive it by employing the method proposed by Dagum (1990). This author explicitly asserts that the social welfare function is a function of income distribution; since the Zenga inequality index also depends on income distribution, his methodology seems to be particularly suitable to our purpose. Dagum (1990) begins his work recalling two principles that are fundamental in any social welfare and income inequality analysis: the aversion to inequality and the aversion to poverty. Describing the second principle, Dagum (1990) takes up an important remark underlined by Sen (1974) and observes that, being the social welfare also a function of the whole income distribution, it cannot be said that social welfare level definitely increases when the average income increases. More specifically, he states that social welfare level might not increase when the income increase takes place in the top part of the income distribution; therefore, the inequality effect would dominate the income effect. Given the Zenga inequality index definition, the Zenga social welfare function strictly depends on the income distribution, so that this function is perfectly able to catch interrelationship between these two effects. By employing the Zenga social welfare function, we show that one can go beyond Dagum's remarks: when the income increase takes place in the top part of the income distribution, the social welfare can decrease when mean income increases. This is our second result.

It is known, however, that the Welfarist approach to the analysis of income distribution is not the only one: to investigate on inequality, an alternative concept may be employed: the concept of deprivation. We conclude the theoretical part of our analysis by shortly evaluating how the Zenga index can be involved in measuring deprivation. To pursue this goal, we propose an alternative way to interpret this index by decomposing it as weighted sum of two components; we also underline that one of them is strictly linked to the Bonferroni index. This is a further result presented in the paper.

The paper is organized as follows. Section 2 presents Gini, Bonferroni and Zenga indices both in their conventional expression and as weighted sum of incomes. Section 3 considers marginal income changes and compares the behaviour of the three indexes. Section 4 deals with the Transfer Sensitivity and the Positional Transfer Sensitivity Principle. Section 5 defines the three social welfare functions, and shows how each index evaluates social welfare gains and losses. Section 6 very briefly focuses on deprivation when the Zenga index is considered. Section 7 presents empirical results, whilst Section 8 offers a conclusion.

2. The Zenga inequality index and its comparison with the Bonferroni and Gini indexes

In this Section we present the Zenga (2007) index and analyse its peculiar features. We then compare it with the Gini and Bonferroni indexes.

Given an income distribution $\{x_1, x_2, ..., x_i, ..., x_N\}$, $x_i \le x_j$, $i < j$, the Zenga index is defined as

$$Z = \frac{1}{N}\sum_{i=1}^{N}\frac{M(x_i) - m(x_i)}{M(x_i)} = 1 - \frac{1}{N}\sum_{i=1}^{N}\frac{m(x_i)}{M(x_i)}, \qquad (1)$$

where $m(x_i)$ is the average for the sub-distribution $\{x_1, x_2, ..., x_i\}$, $(i = 1, 2, ..., N)$,

$$m(x_i) = \frac{1}{i}\sum_{j=1}^{i} x_j \ , \qquad (2)$$

and $M(x_i)$ is the average for the sub-distribution $\{x_{i+1}, x_{i+2}, ..., x_N\}$

$$M(x_i) = \frac{1}{N-i}\sum_{j=i+1}^{N} x_j \ . \qquad (3)$$

It can easily be seen that Z is quite different from most other commonly used inequality measures. First, it is the mean of the ratios between two income functions; second, the denominators of these ratios are not constant. In his seminal article, Zenga (2007) shows that his index takes the value 0 in absence of inequality and the value $\left(1 - \frac{1}{N^2}\right)$ in the case of maximum inequality; as a consequence, $Z = 1$ for $N \to \infty$.

Zenga (2007) also shows that Z meets the Pigou–Dalton transfer principle and it is invariant for proportional changes of all incomes; therefore, it is a relative inequality index. Moreover, Z decreases if a same constant quantity is added to each income within the distribution.[1] Z also satisfies the postulate of symmetry, but, similarly to the Bonferroni index presented below, it does not meet the principle of population invariance.

To better illustrate the characteristics of the Zenga index, we compare it with both the Gini and the Bonferroni indexes. We chose the Gini index (Gini, 1914; Pietra, 1915) because it is the most widely

[1] There is a growing body of literature on the Zenga index and curve. For example, a strand of the literature has focused on the properties of this index (see Polisicchio, 2008; Polisicchio and Porro, 2009; Maffenini and Polisicchio, 2014; Arcagni and Porro, 2014), while other works have focused on inferential theorems and applications (see Greselin and Pasquazzi, 2009; Greselin et al., 2010; Antal et al., 2011; Langel and Tillé, 2012; Greselin et al., 2013, 2014, Ćwiek and Trzcińska, 2023), and still others on the decomposition of the index by population subgroups (Radaelli, 2008, 2010) and by income sources (see Zenga et al., 2012; Pasquazzi and Zenga, 2018). There are also papers focusing on longitudinal decompositions (Mussini, 2013), and applications to real data (see, among others, Arcagni 2013; Jedrzejczak and Trzcinska, 2019). The Zenga index and curve have recently been applied to measure tax redistribution and progressivity (Greselin et al., 2021).

known and used inequality index, and we chose the index proposed by Bonferroni (1930) because it is linked to the Zenga index as we show in Section 6.

Following the suggestions by Sen (1974) about the Gini index,[2] we represent the three indexes as weighted sum of incomes. These representations allow us to illustrate the different role played by the weights associated with each income in evaluating the three indexes. This approach is particularly suitable when the marginal changes in each index value due to a marginal increase of the income unit x_i have to be evaluated (see Section 3).

We begin our analysis by proposing the expression for *Z*. Labelling the mean value of the whole distribution as $\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$, we write:

$$Z = \frac{1}{N\mu}\left(N\mu - \sum_{i=1}^{N} \frac{1}{i}\frac{\mu}{M(x_i)}\sum_{j=1}^{i} x_j \right) = \frac{1}{N\mu}\left(N\mu - \sum_{i=1}^{N} x_i \sum_{j=i}^{N} \frac{1}{j}\frac{\mu}{M(x_j)} \right)$$

$$= \frac{1}{N\mu}\sum_{i=1}^{N} x_i \left(1 - \sum_{j=i}^{N} \frac{1}{j}\frac{\mu}{M(x_j)} \right) = 1 - \frac{1}{N}\sum_{i=1}^{N} x_i \left(\sum_{j=i}^{N} \frac{1}{j}\frac{1}{M(x_j)} \right). \tag{4}$$

Equation (4) underlines that the weights associated with incomes are quite peculiar in the Zenga index, and this feature[3] contributes to making *Z* quite different from the other two inequality indexes. In particular, the presence of the upper mean $M(x_j)$ introduces, through income weights, aspects of the income distribution involving other income earners. From Equation (4) we observe that the term $\sum_{j=i}^{N} \frac{1}{j}\frac{\mu}{M(x_j)}$ decreases as income rank *i* increases, so that $\frac{1}{N\mu}\left(1 - \sum_{j=i}^{N} \frac{1}{j}\frac{\mu}{M(x_j)} \right)$ increases with *i*. We stress, however, that not only the rank *i* plays a role in the behaviour of the weight function, but also the upper mean concurs in determining its expression. It follows that the weights depend on both the income ranks and the inequality characterising the income distribution.

The *Z* weights behaviour may be summarised as follows. They are negative in the bottom percentiles, and they become positive before the 37th percentile: in fact the component $\left(1 - \sum_{j=i}^{N} \frac{1}{j} \right)$, which is negative in the bottom part of the income distribution and increases as *i* increases, becomes positive around the 37th percentile; as $\frac{\mu}{M(x_N)} \leq 1$, the turning point must occur before, and in any case not

[2] Mehran (1976) was the first to highlight the linear structure of the index and its implicit weighting scheme by assigning a particular weight to an individual according to his ranking in the income distribution. Sen (1974) expressed the Gini coefficient as a weighted sum of incomes. Here we employ a rearranged expression of the one originally proposed by Sen.

[3] This behaviour is a consequence of being the index, considered in its original expression, an average of ratios between two income functions.

after, the 37th percentile. The turning point thus depends on the inequality in the income distribution, which is reflected in the factor $\frac{\mu}{M(x_i)}$. Moreover, Imedio-Olmedo et al. (2013) stress that $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=i}^{N}\frac{1}{j}=1$; it follows that $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=i}^{N}\frac{\mu}{jM(x_N)}<1$: it is equal to 1 only if $x_i=x_j, \forall\, i$. In the remainder of this Section, we compare Equation (4) with the corresponding ones obtained for the Bonferroni and Gini indexes.[4]

The Bonferroni index, here denoted by B, can be evaluated as follows:

$$B=\frac{1}{N}\sum_{i=1}^{N}\frac{\mu-m(x_i)}{\mu}. \tag{5}$$

To evaluate the income weights, Equation (5) can be rewritten as

$$B=\frac{1}{N\mu}\left(N\mu-\sum_{i=1}^{N}\frac{1}{i}\sum_{j=1}^{i}x_j\right)=\frac{1}{N\mu}\left(N\mu-\sum_{i=1}^{N}x_i\sum_{j=i}^{N}\frac{1}{j}\right)=\frac{1}{N\mu}\sum_{i=1}^{N}x_i\left(1-\sum_{j=i}^{N}\frac{1}{j}\right)$$

$$=1-\frac{1}{N\mu}\sum_{i=1}^{N}x_i\sum_{j=i}^{N}\frac{1}{j}. \tag{6}$$

Equation (6) shows that the income weights $\sum_{j=i}^{N}\frac{1}{j}$ decrease when the income rank i increases; therefore, the weights $\frac{1}{N\mu}\left(1-\sum_{j=i}^{N}\frac{1}{j}\right)$ increase, which is analogous to what happens for the Zenga index. As we have already observed, the weights are negative at the initial percentiles, they increase with i, and become positive around the 37th percentile; their maximum, associated with x_N, is $\frac{1}{N\mu}\left(1-\frac{1}{N}\right)$.

These behaviours associated with the Bonferroni index are similar to those described for the Zenga index, even if in this latter case the weights become positive before the 37th percentile. However, it has to be noted that the weights in Equation (5) are only a function of the rank i of the income x_i; they are affected neither by incomes greater than x_i, nor by any other characteristic of the distribution except, obviously, by the overall average μ. As a consequence, unlike the Zenga index, when the Bonferroni index is considered, the turning point percentile does not depend on the characteristics of the distribution.

Let us now consider the Gini index:

[4] For a comparison of the Gini and Bonferroni indexes, see also de Vergottini (1940).

$$G = \frac{1}{2\mu N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|x_i - x_j\right| = \frac{1}{N\mu}\sum_{i=1}^{N} x_i\left(2\frac{i}{N} - 1 - \frac{1}{N}\right). \tag{7}$$

From Equation (7) it is easy to see that the weight $\frac{1}{N\mu}\left(2\frac{i}{N}-1-\frac{1}{N}\right)$, associated with each income, depends only on the income rank *i*. Their behaviour is linear, contrary to what occurs in the Bonferroni and Zenga indexes, for which the weights are non-linear.

As $\frac{1}{N}$ is negligible if *N* is large, we can write $\left(2\frac{i}{N}-1-\frac{1}{N}\right) \cong \left(2\frac{i}{N}-1\right)$, which is negative up to the median and positive thereafter; this implies that the contribution of the income x_i is negative up to the median and positive thereafter. Similar to the Bonferroni index, and unlike the Zenga index, the turning point percentile does not depend on the income distribution.

3. Derivatives: Comparing the behaviour of the three indexes

Here we focus on the partial derivative with respect to income x_i for all indexes. Looking at Equations (4), (6) and (7), the effect that a change in the individual *i*'s income has on *Z*, *B* and *G* can be derived. It is worth pointing out that we consider income changes that do not alter the ranking of the income parade.

For the Gini index, the partial derivative with respect to x_i is

$$\frac{\partial G}{\partial x_i} = \frac{1}{N\mu}\left(2\frac{i}{N}-1-\frac{1}{N}\right) - \frac{1}{N^2\mu^2}\sum_{i=1}^{N} x_i\left(2\frac{i}{N}-1-\frac{1}{N}\right)$$

$$= \frac{1}{N\mu}\left(2\frac{i}{N}-1-\frac{1}{N}-G\right). \tag{8}$$

From Equation (8), we notice first that the index derivative, $\frac{\partial G}{\partial x_i}$, is linear; moreover, it depends on both the position of the individual *i* in the income parade and the starting value of *G*: given the position *i*, *G* shifts the derivative upward or downward. Negative derivatives occur up to the median plus $\frac{G}{2}$; in comparing two distributions characterised by different degrees of inequality, according to the position in which the income increase takes place, inequality increases are lower and, consequently, reductions in inequality are greater where *G* is greater.

Looking at the Bonferroni index, the partial derivative with respect to x_i is:

$$\frac{\partial B}{\partial x_i}=\frac{1}{N\mu}\left(-\sum\nolimits_{j=i}^{N}\frac{1}{j}\right)+\frac{1}{N^2\mu^2}\sum\nolimits_{i=1}^{N}x_i\sum\nolimits_{j=i}^{N}\frac{1}{j}=\frac{1}{N\mu}\left(-\sum\nolimits_{j=i}^{N}\frac{1}{j}+\frac{1}{N\mu}\sum\nolimits_{i=1}^{N}x_i\sum\nolimits_{j=i}^{N}\frac{1}{j}\right)$$

$$=\frac{1}{N\mu}\left[-\sum\nolimits_{j=i}^{N}\frac{1}{j}+(1-B)\right]. \tag{9}$$

In Equation (9) the partial derivative also depends on both the position i and the starting value of the index B. For a given position i, Equation (9) also shows that different values of B only shift the partial derivative upward or downward. However, in the case of the Bonferroni index, when N is sufficiently large, negative partial derivatives occur approximately up to the $\exp\{-1+B\}^{\text{th}}$ percentile. Unlike the Gini index, in this case the derivative is non-linear and smooths as i increases. Excluding these peculiarities, considerations similar to those made for the derivative of G apply to the derivative of B.

In the case of the Zenga index, the partial derivative is more complex:

$$\frac{\partial Z}{\partial x_i}=-\frac{1}{N}\sum\nolimits_{j=i}^{N}\frac{1}{j}\frac{1}{M\left(x_j\right)}+\frac{1}{N}\sum\nolimits_{h=1}^{i-1}x_h\sum\nolimits_{j=h}^{i-1}\frac{1}{j(N-j)}\frac{1}{M\left(x_j\right)^2}. \tag{10}$$

In the right hand side of Equation (10), the first summation is negative, as it involves the addition of positive summands, which diminish in number and in value as i increases, multiplied by (–1); the second summation is positive, as it is the addition of positive summands – except the first one, which is zero – that increase in number as i increases. Consequently, the more i increases, the more $\left(\partial Z/\partial x_i\right)$ increases as well, although not at same rate: due to the combined effect of the two addends, the values of $\left(\partial Z/\partial x_i\right)$ turn step-by-step from negative to positive.

In the Bonferroni case (Equation 9), the derivatives increase at a decreasing rate with respect to i, so they are represented by concave lines. Moreover, in both the Gini and Bonferroni cases, the partial derivative curves do not intersect, and the curve of the lowest inequality case dominates the corresponding curves associated with higher inequality levels. Therefore, if $G_1 < G_2$ and $B_1 < B_2$, since $\frac{\partial G_1}{\partial x_i} > \frac{\partial G_2}{\partial x_i}$ and $\frac{\partial B_1}{\partial x_i} > \frac{\partial B_2}{\partial x_i}$, in the bottom part of the income distribution, characterized by negative derivatives, $\left|\frac{\partial B_1}{\partial x_i}\right| < \left|\frac{\partial B_2}{\partial x_i}\right|$ is observed, whilst in the top part of the income distribution, characterized by positive derivatives, we get $\frac{\partial G_1}{\partial x_i} > \frac{\partial G_2}{\partial x_i}$ and $\frac{\partial B_1}{\partial x_i} > \frac{\partial B_2}{\partial x_i}$.

On the contrary, in the case of the Zenga index the curves do in fact intersect. In particular, the higher the level of inequality, the higher the curves up to approximately to the median; afterwards the situation is reversed, so that the lower the inequality level, the lower the curves. As a matter of fact, in the Zenga index case, if $Z_1 < Z_2$, $\left|\frac{\partial Z_1}{\partial x_i}\right| > \left|\frac{\partial Z_2}{\partial x_i}\right|$ is observed for the negative part, whilst $\frac{\partial Z_1}{\partial x_i} > \frac{\partial Z_2}{\partial x_i}$ for the positive part. From these considerations comes the fact that the curves do intersect; moreover, simulations (see Section 7) show that the greater the Zenga index the derivative lines are all the more flattened towards the x-axis.

4. Transfer sensitivity

In this Section we consider how the three indexes react to a marginal transfer *dx* from x_s to x_i, with $i < s$, and analyse the behaviour of the total differential. The comparison among the three indexes suggests investigating another aspect of I_Z – that is, its transfer sensitivity. It is known that in addition to the above-cited principles that an inequality index must satisfy, there are other postulates that an index may or may not satisfy. These include positional transfer sensitivity, which implies that, in transferring a same amount from a richer recipient to a poorer one, the reduction in inequality is greater the lower the rank of the recipient is. The Gini index does not satisfy this principle, while the Bonferroni index does.

Let us begin by considering how the Gini index reacts to a marginal transfer *dx* from x_s to x_i, $i < s$. From Equation (8) we easily obtain that the effect of a marginal transfer is

$$dG = \frac{\partial G}{\partial x_i}dx - \frac{\partial G}{\partial x_s}dx = \frac{2(i-s)}{N^2\mu}dx\,. \qquad (11)$$

From Equation (11), we see immediately that, because *dG* is negative, the index respects the condition required to satisfy the Pigou-Dalton transfer principle. Moreover, the differential in Equation (11) depends only on the difference between ranks and is proportional to it; therefore, the Gini index is sensitive to income transfer in the same way, independently of the position of the two units involved in the transfer. In other words, this means that the index fails to demonstrate positional transfer sensitivity – or, it is worth pointing out, that the only characteristics of the income distribution that the transfer effect reflects are the income average and the number of incomes.

For the *B* index, from Equation (9) we obtain

$$dB = \frac{\partial B}{\partial x_i}dx - \frac{\partial B}{\partial x_s}dx = -\frac{1}{N\mu}\sum_{h=i}^{s-1}\frac{1}{h}dx\,, \qquad (12)$$

which is negative, as expected for the Pigou-Dalton transfer principle to be satisfied. We can observe that the differential *dB* not only depends on the difference between *s* and *i*, but also on the position of the two incomes x_i and x_s in the income distribution $\{x_1, x_2, ..., x_i, ..., x_N\}$. In the Bonferroni index, as Chakravarty and Muliere (2003, p. 471) stress, "*A progressive transfer is valued more if the transfer occurs at lower income levels*". Unlike the Gini index, the Bonferroni index satisfies the positional sensitivity transfer principle: the higher the percentiles of two units, the lesser the effect in diminishing the index value – that is, in diminishing inequality. According to *dB*, as in Equation (12), the only characteristics of the income distribution that the transfer effect reflects are the income average and the number of incomes. It follows that the changes *dG* and *dB* depend on the same two variables, which are completely independent from the behaviour of the income distribution under consideration.

It is much more complex to anlyse the effect of a *Robin Hood* (i.e. an egalitarian transfer) transfer in the case of the Zenga index. From Equation (10), we obtain the differential *dZ*:

$$dZ = \frac{\partial Z}{\partial x_i}dx - \frac{\partial Z}{\partial x_s}dx = -\frac{1}{N\mu}\sum_{j=i}^{s-1}\frac{1}{j}\frac{\mu}{M(x_j)}dx - \frac{1}{N\mu}\sum_{h=i}^{s-1}x_h\sum_{j=h}^{s-1}\frac{1}{j(N-j)}\frac{\mu}{M(x_j)^2}dx. \tag{13}$$

The Zenga index meets the Pigou-Dalton transfer principle, as, in case of an egalitarian transfer, *dZ* is negative. As in Equations (11) and (12), we can immediately observe that Equation (13) is greater the greater the distance in the rank between *i* and *s*; however, given the average μ and the number of units *N*, it depends not only on the rank of the two units involved, but also on the characteristics of the distribution reflected by $M(x_j)$.

As we will see (Section 7), simulations show that, when the difference *i*–*s* decreases, both *dG* and *dB* decrease in absolute value. In particular, the relationship is linear for the Gini index and concave for the Bonferroni index, and the relationship is the same whatever the degree of income inequality. In the case of the Zenga index, the relationship is concave but differs according to the level of income inequality. If the difference *i*–*s* is invariant, *dG* does not vary, while *dB* decreases, and this behaviour is the same for all income distributions. In the Zenga case, *dZ* first increases, then it is constant, and, when *i*–*s* involves the richest individuals, *dZ* decreases.

5. Contribution to social welfare gain and loss

Following the approach developed by Dagum (1990), Equations (4), (6) and (7) are also particularly suitable for yielding the Social Welfare levels related to the Zenga, Bonferroni and Gini indexes, respectively. Dagum (1990) highlights the relationship between an income inequality measure and the social welfare function: he shows that an inequality measure can be inferred from a Social Welfare function (henceforth *SW*) and a *SW* can be deduced from an income inequality measure.

Dagum states that the overall average income μ can be split into the sum of the average social welfare gain $SW\{\mathbf{x}\}$ and loss $LOSS\{\mathbf{x}\}=\mu I\{\mathbf{x}\}$, where $I\{\mathbf{x}\}$ is a proper inequality index defined on the interval 0–1.[5] According to Dagum (1990), each income in the income vector can be split into the sum of two components that represent the contribution of that particular income to the gain and to the loss of social welfare. We can therefore write:

$$x_i = x_i \cdot w_i\left\{\mathbf{x}\right\} + x_i \cdot \bar{w}_i\left\{\mathbf{x}\right\}, \tag{14}$$

where $x_i \cdot w_i\left\{\mathbf{x}\right\}$ is the contribution of x_i to $SW\left\{\mathbf{x}\right\}$ and $x_i \cdot \bar{w}_i\left\{\mathbf{x}\right\}$ is the contribution to $LOSS\left\{\mathbf{x}\right\}$. From Equation (14), it is easy to see that the sum of the two weights is equal to one; moreover, both average social welfare loss and average social welfare gain can be written as

$$LOSS\left\{\mathbf{x}\right\} = \mu I\left\{\mathbf{x}\right\} = \frac{1}{N}\sum_{i=1}^{N} x_i \cdot \bar{w}_i\left\{\mathbf{x}\right\} \text{ and } SW\left\{\mathbf{x}\right\} = \frac{1}{N}\sum_{i=1}^{N} x_i \cdot w_i\left\{\mathbf{x}\right\}, \tag{15}$$

$$w_i\left\{\mathbf{x}\right\} \text{ and } \bar{w}_i\left\{\mathbf{x}\right\} \geq 0,\ w_i\left\{\mathbf{x}\right\} + \bar{w}_i\left\{\mathbf{x}\right\} = 1.$$

From Equation (15) it follows that $\mu = SW\left\{\mathbf{x}\right\} + LOSS\left\{\mathbf{x}\right\}$ and that, if $LOSS\left\{\mathbf{x}\right\} = \mu I\left\{\mathbf{x}\right\}$, $SW\left\{\mathbf{x}\right\} = \mu\left(1 - I\left\{\mathbf{x}\right\}\right)$.[6]

Considering the Gini index as in Equation (7), the weight $\bar{w}_i\left\{\mathbf{x}\right\}$ is

$${}_G\bar{w}_i\left\{\mathbf{x}\right\} = \left(2\frac{i}{N} - 1 - \frac{1}{N}\right), \tag{16}$$

from which we can immediately derive the social welfare weights

$${}_G w_i\left\{\mathbf{x}\right\} = 1 - {}_G\bar{w}_i\left\{\mathbf{x}\right\} = \left(2 - 2\frac{i}{N} + \frac{1}{N}\right). \tag{17}$$

More specifically, the social welfare function is

$${}_G SW\left\{\mathbf{x}\right\} = \frac{1}{N}\sum_{i=1}^{N} x_i \cdot \left(2 - 2\frac{i}{N} + \frac{1}{N}\right). \tag{18}$$

For the Bonferroni index, as in Equation (6), the weights that enter Equation (15) are

$${}_B\bar{w}_i\left\{\mathbf{x}\right\} = 1 - \sum_{j=i}^{N} \frac{1}{j} \text{ and } {}_B w_i\left\{\mathbf{x}\right\} = \sum_{j=i}^{N} \frac{1}{j}, \tag{19}$$

so that

$${}_B SW\left\{\mathbf{x}\right\} = \frac{1}{N}\sum_{i=1}^{N} x_i \cdot \sum_{j=i}^{N} \frac{1}{j}. \tag{20}$$

Finally, considering the Zenga index, from Equation (4), we can derive

[5] Dagum (1990), Equation (6), page 94. Here we decline Dagum's analysis and results in the discrete case.

[6] Presenting this result Dagum observes that "it is also given by Atkinson (1970) in relation to his inequality measure and by Blackorby and Donalson (1978). None of these authors, however, deduce it from the economic units' social utility and disutility functions." (Dagum 1990, p. 94). Social utility and disutility are defined as gain and loss of social welfare.

$$ {}_Z\bar{w}_i\{\mathbf{x}\} = 1 - \sum_{j=i}^{N} \frac{1}{j} \frac{\mu}{M(x_j)} \text{ and } {}_Z w_i\{\mathbf{x}\} = \sum_{j=i}^{N} \frac{1}{j} \frac{\mu}{M(x_j)}. \tag{21} $$

From Equation (21), we obtain

$$ {}_Z SW\{\mathbf{x}\} = \frac{1}{N} \sum_{i=1}^{N} x_i \cdot \sum_{j=i}^{N} \frac{1}{j} \frac{\mu}{M(x_j)}. \tag{22} $$

We stress that in the case of both the Gini index (Equations (16) and (17)) and the Bonferroni index (Equation (19)), we could simply write ${}_G w_i\{\mathbf{x}\}$, ${}_B w_i\{\mathbf{x}\}$, ${}_G\bar{w}_i\{\mathbf{x}\}$ and ${}_B\bar{w}_i\{\mathbf{x}\}$ as ${}_G w_i$, ${}_B w_i$, ${}_G\bar{w}_i$ and ${}_B\bar{w}_i$, respectively, because these functions depend only on the position of x_i in the income parade. In both cases, ${}_G w_i$, ${}_B w_i$, ${}_G\bar{w}_i$ and ${}_B\bar{w}_i$, are positive: the former are a decreasing function of rank, while the latter are increasing. In the case of the Gini index, w_i and $\bar{w}_i$ are linear, while in the case of the Bonferroni index, they are non-linear functions of rank.[7] In Equation (21), ${}_Z w_i\{\mathbf{x}\}$ and ${}_Z\bar{w}_i\{\mathbf{x}\}$ depend not only on the position of x_i in the income parade, but also on the vector **x**; more precisely, they depend on the average the sub-distribution $\{x_{i+1}, x_{i+2}, ..., x_N\}$.

As Dagum (1990) stresses, to evaluate gain and loss of welfare, an individual can take into account not only one's own income, but also the incomes of other components of society, "*in particular of those with income greater than his own and the frequency of economic units with income greater or lower than his own*". Equation (21) fulfils this remark. In the case of *G* and *B*, given Equations (18) and (20), an increase in the income of whatever income unit always generates an increase in ${}_G SW\{\mathbf{x}\}$: the position of the income unit in the income parade only affects the magnitude of the increase of the overall *SW*. In the Gini and Bonferroni indexes, the increase in ${}_G SW\{\mathbf{x}\}$ and ${}_B SW\{\mathbf{x}\}$ depends only on the position of the income unit that benefits from the increase, and it is independent both from the level of x_i and from μ: the only characteristic of the distribution that affects the increase in ${}_G SW\{\mathbf{x}\}$ and ${}_B SW\{\mathbf{x}\}$ is the number of income units.

In the case of the Zenga index, ${}_Z w_i\{\mathbf{x}\}$ and ${}_Z\bar{w}_i\{\mathbf{x}\}$ are non-linear as well. In particular, at the lowest ranks, the ratio $\frac{\mu}{M(x_i)}$ can be not very different from 1, while the more the rank increases, the more the ratio falls: this can make ${}_Z w_i\{\mathbf{x}\}$ flatter for the Zenga index than for the Bonferroni index. It is also important to observe that a change in the *i*-th unit income not only modifies x_i but also modifies

[7] In the continuous case, for the percentile *p*, $w(x_p) = -\ln(x_p)$, which is a decreasing function that has a very high negative slope for the lowest percentiles and almost flat slopes for the highest. See, e.g., Son (2011) and Greselin et al. (2021).

${}_{Z}w_j\{\mathbf{x}\}$ as well as ${}_{Z}\overline{w}_j\{\mathbf{x}\}$, j=1, 2, …, (i-1), due to the modifications of the ratios $\frac{\mu}{M(x_j)}$, which decrease (increase) if x_i increases (decreases). It follows that the final result may be a decrease in the overall ${}_{Z}SW\{\mathbf{x}\}$, instead of an increase.

In the case of the Gini and Bonferroni indexes, we have

$\frac{\partial_G SW\{\mathbf{x}\}}{\partial x_i} = \frac{1}{N}\left(2-2\frac{i}{N}+\frac{1}{N}\right)$ and $\frac{\partial_B SW\{\mathbf{x}\}}{\partial x_i} = \frac{1}{N}\sum_{j=i}^{N}\frac{1}{j}$, respectively, which are positive and monotonically decreasing with respect to i, and uniquely depend on i.

For the Zenga index, however – given Equations (4) and (22) – we get

$$\frac{\partial_Z SW\{\mathbf{x}\}}{\partial x_i} = \frac{1}{N}(1-I_Z) + \frac{\mu}{N}\sum_{j=i}^{N}\frac{1}{j}\frac{1}{M(x_j)} - \frac{\mu}{N}\sum_{h=1}^{i-1}x_h\sum_{j=h}^{i-1}\frac{1}{j(N-j)}\frac{1}{M(x_j)^2}. \tag{23}$$

In Equation (23), which is not a function of rank alone, the first addend is constant and positive; the second one is also positive, but it is a decreasing function of i. Conversely, the third one is zero for i=1; for i>1, it is negative and, in absolute terms, ceteris paribus, it is an increasing function of the rank of the unit receiving the increase. This term can significantly affect the derivative shown in Equation (21).

It is trivial to observe that $LOSS\{\mathbf{x}\}$ fulfils the *Aversion to Inequality Principle*, $(LOSS/\mu) = I\{\mathbf{x}\}$ for the three indexes. The part concerning the *Aversion to Poverty Principle* is clearly represented by $SW\{\mathbf{x}\}$: in the Gini and the Bonferroni indexes, ${}_{G}SW\{\mathbf{x}\}$ and ${}_{B}SW\{\mathbf{x}\}$ are increasing functions of μ, whatever unit receives the additional income amount. In the Zenga index as well, ${}_{Z}SW\{\mathbf{x}\}$ represents this second fundamental principle; however, because an additional income is perceived not only by the beneficiary unit (not in all circumstances), an increase in μ does not necessarily generate an increase in ${}_{Z}SW\{\mathbf{x}\}$: in certain circumstances, as the simulations reported in Section 7 show, an increase in the highest percentiles of the distribution can diminish $SW\{\mathbf{x}\}$. Moreover, in the Zenga index case, the effects of an increase depend not only on the position of the income earner receiving the increase, but also on the starting value of Z: in the empirical section simulations show that the lower Z, the stronger the effects.

6. The Zenga index and deprivation

In Section 5, we have presented and discussed the welfare functions associated with the three here considered indexes. It is known, however, that the Welfarist approach to the analysis of income distribution is not the only approach. As observed by Cowell (2008), alternative approaches have attempted to reconsider the fundamental nature of income inequality using concepts that lie outside the familiar territory of social-welfare analysis. Typically, these approaches focus on income differences rather than on individual income level; one such approach is the theory of deprivation. In this Section, we analyse the three indexes from this perspective. The concept of deprivation has its origin in sociology and was introduced by Runciman (1966). Substantially, for any person in the society, the feeling of deprivation arises out of comparison of his or her situation with those of better-off persons.

Yitzhaki (1979) and Hey and Lambert (1980) defined deprivation in terms of income and, starting from two different remarks by Runciman (1966), obtained the same result; they showed that a plausible index of deprivation in a society is the product of the Gini index and the average income – that is, the income that each person would have in an egalitarian society. The Yitzhaki's (1979) starting point is that "*the social evaluation of the deprivation inherent in a person's not having X is an increasing function of the proportion of those do have it*" (p. 321). Yitzhaki observes that the degree of deprivation is the complement to the overall average of the degree of satisfaction. As a consequence, denoting the Gini index with G as usual, with μ the overall mean and with S the overall satisfaction, he writes: $S=\mu(1-G)$. Recalling that ${}_{G}SW\{\mathbf{x}\}=\mu(1-G)$ is the abbreviated welfare function associated with the Gini index,[8] it is easy to see that Yitzhaki (1979) has shown that the same expression can be used to evaluate two different concepts: the satisfaction in a society and its social welfare.[9] From this conclusion, it derives that the quantity μG – which in the previous Section, following the Welfarist prospective of Dagum, we have defined as welfare loss – in deprivation perspective, following Yitzhaki, may be defined as the loss of satisfaction.

This different statement of Runciman's (1966, p.10) was essential to Hey and Lambert (1980) to obtain their result: "*the magnitude of relative deprivation is the extent of the difference between the desired situation and that of the person desiring it*". Their reasoning can be briefly summarised as follows. They define the deprivation sensed by subject i with respect to a richer subject j as the income difference $x_j - x_i$, if $x_j < x_i$, the subject does not feel deprivation, so its value is equal to zero. A

[8] As Creedy (2021, p. 9) pointed out, it is called an abbreviated function because it is expressed in terms of just two variables, the arithmetic mean and the inequality measure, although both of these are of course a function of all income.

[9] For further information on this topic, see Yitzhaki (1982) and Bishop et al. (1989).

measure of the deprivation experienced by i, when comparing his or her income with all other incomes, is given by the average of the differences between his or her income and all other incomes. From this statement it appears that the average deprivation sensed in a society can be evaluated by the product of Gini index and the mean income.

Let us now consider how the Bonferroni index can be involved in measuring deprivation. In so doing, we follow Chakravarty (2007)'s suggestions.[10] As seen above, the approaches of Yitzhaki (1979) and Hey and Lambert (1980) suggest a deprivation measure that springs from pair-wise comparisons of all individual incomes. Chakravarty (2007) looks at deprivation, as expressed by the Bonferroni index, from a different point of view and defines a deprivation measure linked to population groups. In this perspective, given an income distribution represented by a non-decreasing succession of N incomes x_i, we consider as a population group i the set of individuals with incomes lower than x_i and denote $m(x_i)$ the mean income of this group. Because we have N income values, we have N groups. From Equation (6), the product of the Bonferroni index and the population mean μ is

$$\mu B = \frac{1}{N}\sum_{i=1}^{N}\left[\mu - m\left(x_i\right)\right] = \frac{1}{N}\sum_{i=1}^{N} x_i - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{i}\sum_{j=1}^{i} x_j \,. \tag{24}$$

Equation (24) defines μB as the mean of the differences between the mean μ and the partial mean $m(x_i)$; this tells us the amount by which the mean of the mean incomes of the N groups is less than the entire mean μ. It is not difficult to see that Equation (24) rewrites as

$$\mu B = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{i}\sum_{j=1}^{i}\left(x_i - x_j\right). \tag{25}$$

In Equation (25), the difference $(x_i - x_j)$ represents the shortfall of income x_j with respect to the reference value x_i. It can be defined as a measure of the deprivation experienced by the individual j in the i-group. The quantity $\frac{1}{i}\sum_{j=1}^{i}\left(x_i - x_j\right)$ is then the average shortfall of the i units having an income less than or equal to x_i. This can be interpreted as an indicator of the average deprivation sensed by the i- group. It follows that μB, as in Equation (25), is the average of the N average group shortfalls, each of which is associated with an income level x_i. This means that μB may be seen as the mean of the average deprivation of each group.

Turning to the Zenga index, we reconsider the product of this index and the average income that, in the previous Section, was defined as welfare loss. Our goal is now to analyse how the same quantity μZ can be interpreted as satisfaction loss or, better still, as a plausible index of deprivation. To pursue

[10] Imedio-Olmedo et al. (2012) showed that the deprivation measure derived from the Bonferroni index can be obtained by an approach analogous to that discussed in Hey and Lambert (1980); however, they identify the level of subject i with $m(x_i)$.

this objective, we propose here an alternative expression for the Zenga index by decomposing it as a sum of two components. Adding and subtracting x_i in its expression, μZ can be rewritten as follows:

$$\mu Z = \frac{\mu}{N}\sum_{i=1}^{N}\frac{M(x_i)-m(x_i)}{M(x_i)} = \frac{1}{N}\sum_{i=1}^{N}\left[M(x_i)-x_i\right]\frac{\mu}{M(x_i)} + \frac{1}{N}\sum_{i=1}^{N}\left[x_i - m(x_i)\right]\frac{\mu}{M(x_i)}. \qquad (26)$$

Let us focus on the second component in Equation (26). Following the suggestions of Chakravarty (2007), we write the quantity within the square bracket as a sum of differences between x_i (i=1, 2,…, N) and all incomes less than or equal to it – that is, $x_i - m(x_i) = \frac{1}{j}\sum_{j=1}^{i}\left[x_i - x_j\right]$. This expression is the average of the shortfalls of the incomes less than or equal to x_i with respect to x_j and, as said before, it is a measure of the average deprivation experienced by the i-group. Consequently, the second component can be rewritten as

$$\frac{1}{N}\sum_{i=1}^{N}\left[x_i - m(x_i)\right]\frac{\mu}{M(x_i)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{i}\frac{\mu}{M(x_i)}\sum_{j=1}^{i}\left(x_i - x_j\right). \qquad (27)$$

The link of Equation (27) with the product of the Bonferroni index and the distribution mean as in Equation (25) is quite clear. In Equation (27), the average deprivation of each group is weighed by the factors $\mu/M(x_i)$, which decreases as i increases.[11] Equation (27) is, then, the overall average of the N weighed average deprivation of the groups. This weighting system implies that when the income benchmark x_i increases and the group includes an increasing number of incomes, its weight decreases; therefore, the contribution of the average deprivation of the group to the overall average deprivation decreases. This means that this component of the Zenga index gives more importance to the average shortfalls calculated among lower incomes than among higher incomes. In short, this weight system implies that moving up in the rank of the distribution, the sensitivity to the effects of the differences decreases, and this happens the greater the inequality in the distribution.

Turning to the first addend in Equation (27), we write

$$\frac{1}{N}\sum_{i=1}^{N}\left[M(x_i)-x_i\right]\frac{\mu}{M(x_i)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{N-i}\frac{\mu}{M(x_i)}\sum_{j=i+1}^{N}\left[x_j - x_i\right]. \qquad (28)$$

In this addend, each individual compares his or her situation with that of all of the individuals who are richer than he or she is. The quantity $\frac{1}{N-i}\frac{\mu}{M(x_i)}\sum_{j=i+1}^{N}\left[x_j - x_i\right]$ in Equation (28) is, then, the average shortfall of the i-th individual with respect to the $N-i$ individuals who are richer than he (or she) is, and it can be interpreted as a measure of the feeling of deprivation experienced by the

[11] We underline that the behaviour of the weights is linked to the inequality of the income distribution.

individual i and, as in Equation (27), the average shortfall is weighed by the factor $\mu/M(x_i)$. Exactly as happens in the first addend, the more the income of unit i increases, the less importance the weight system attaches to the measure of its deprivation. Equation (28) is the overall average of N average shortfalls, and may be interpreted as the average of the feelings of deprivation experienced by all of the individuals when they compare themselves with that part of the population that is richer than they are.

Unlike the Bonferroni index, which takes into account only comparisons with lower incomes, μZ can then be split into the sum of two aspects of social deprivation. The first aspect springs from individuals identifying themselves with those with lower incomes and, in calculating the overall average deprivation, comparisons with higher incomes are ignored. In the second aspect, on the contrary, the social average deprivation is the result of comparisons with higher incomes, and the lower incomes are ignored. An appropriate system of weights acts as a glue to these two aspects when calculating the average measure of social deprivation.

7. Empirical investigation

7.1. Data

To evaluate the behaviour of the formulas related to the three inequality indexes, we employ several lognormal distributions characterised by the same equal mean income. This empirical strategy can show how different shapes in the income distributions influence the trend in which we are interested, unlike the use of microsimulation models.

In particular, we consider 23 lognormal distributions $\boldsymbol{X}$ of 1,000 income earners each; a random income is associated with each income earner.[12] The log-transformed distributions are characterised by a zero mean and a standard deviation SD ranging from 0.3 to 1.4. We then adjust income values to obtain a mean value equal to exactly 20,000 euros. Moreover, because we are also dealing with income transfers, we operate them by employing the same income amount able not to generate re-ranking in all income distributions employed. Table 1 shows the three inequality indexes for our 23 distributions. As it is well known, given a mean income, inequality increases as the standard deviation (SD) increases: the Gini coefficient (Gini, 1914; Pietra, 1915) increases from 0.166 when SD=0.3 is considered up to 0.688 when SD=1.4 is considered. The same happens for both the Bonferroni and the Zenga indexes: when SD=0.3, the Zenga index is equal to 0.414, and it increases up to 0.929 when SD=1.4 is considered; for the Bonferroni index, it increases from 0.242 to 0.785, respectively.

[12] Employing the STATA command “rndlgn”.

Table 1: Inequality indexes for all lognormal distributions

SD	Gini	Zenga	Bonferroni
0.30	0.16650	0.41379	0.24198
0.35	0.19642	0.46723	0.27953
0.40	0.21461	0.50051	0.30538
0.45	0.25294	0.56025	0.35273
0.50	0.28537	0.60494	0.39023
0.55	0.30149	0.62622	0.40988
0.60	0.32958	0.66210	0.44191
0.65	0.35093	0.68917	0.46886
0.70	0.36849	0.70709	0.48661
0.75	0.39694	0.73647	0.51732
0.80	0.44261	0.77850	0.56357
0.85	0.44803	0.78167	0.56761
0.90	0.47291	0.80457	0.59559
0.95	0.50611	0.82834	0.62547
1.00	0.51101	0.83242	0.63171
1.05	0.51387	0.83791	0.63820
1.10	0.55802	0.86457	0.67581
1.15	0.60269	0.88788	0.71277
1.20	0.59916	0.88762	0.71163
1.25	0.61857	0.89828	0.72868
1.30	0.63530	0.90668	0.74300
1.35	0.64653	0.91273	0.75331
1.40	0.68837	0.92867	0.78477

Source: Own elaborations.

7.2. Inequality curves

We can start by plotting the Lorenz, Bonferroni and Zenga curves. Given an income distribution $\{x_1, x_2, ..., x_i, ..., x_N\}$, $x_i \leq x_j$, $i < j$, we define the abscissa values for all the three curves as $p_i = \frac{i}{N}$.

For the Lorenz curve, each ordinate is defined as $L_i = \frac{\sum_{i=1}^{j} x_i}{\sum_{i=1}^{N} x_i}$. The line of perfect equality is the diagonal line, in which $L_i = p_i$ for all *i*; the curve of perfect inequality (all individuals earn a zero income but one who earns a positive income), meanwhile, is the line parallel to the x-axis when the ordinate is equal to 0 for the bottom $N-1$ income earners; when the individual with positive income is also considered, then $p_i = 1$ and $L_i = 1$ (Pellegrino, 2020).

For the Bonferroni curve, each ordinate is equal to $B_i = \frac{L_i}{p_i} = 1 - \frac{\mu - m(x_i)}{\mu} = \frac{m(x_i)}{\mu}$. In the case of perfect inequality, the Bonferroni curve is equal to the Lorenz curve; in the case of perfect equality, it is the line parallel to the x-axis when the ordinate is equal to 1.

Finally, for the Zenga approach, the curve analogous to the Bonferroni curve is what Zenga (2007) calls the Uniformity curve,[13] those ordinates are defined as $U_i = \frac{m(x_i)}{M(x_i)}$.

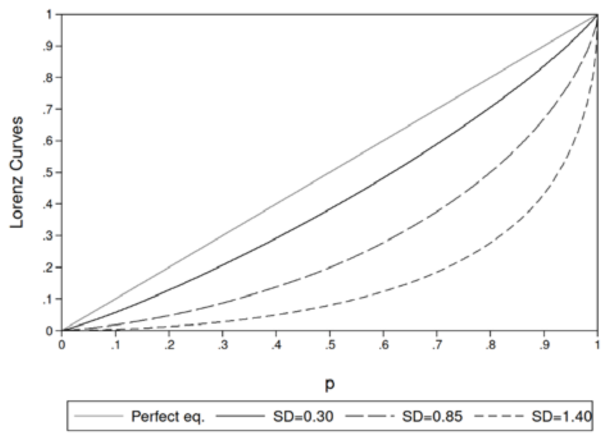
In case of perfect inequality – that is, when $N-1$ individuals have zero income and one individual possesses the whole amount – $L_i \equiv B_i \equiv U_i$.

The Zenga curve is equal to both the Lorenz and the Bonferroni curves when all of the $N-1$ individuals with zero income are considered; when all N individuals are considered, $L_i = B_i = 1$, while $U_i < 1$, and its value depends on the income level of the richest individual. Finally, in case of perfect equality, the Zenga curve is the line parallel to the x-axis when the ordinate is equal to 1, as in the Bonferroni curve.

Figures 1, 2 and 3 plot the Lorenz curve, the Bonferroni curve and the Zenga curve, respectively. Here we consider only three curves according to the lowest, highest and median value of the standard deviations we considered (0.30, 0.85 and 1.40).

As usual, the Lorenz curves identifies the percentage of total income accruing to the bottom p_i percent of income earners, once their incomes are sorted in non-decreasing order. For example, let consider the Lorenz curve when SD=0.85 (Figure 1).

Figure 1: Lorenz curves when SD varies

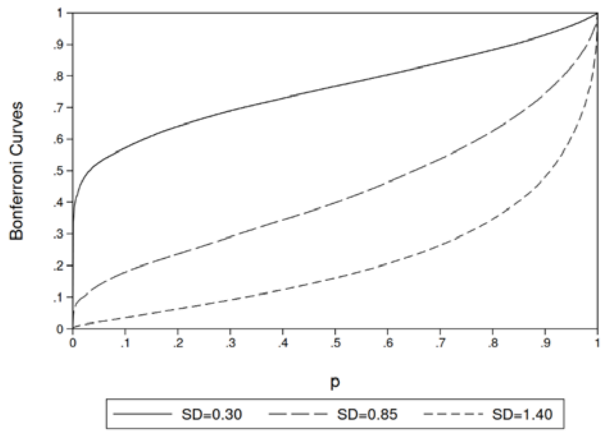


[13] Zenga (2007) also considers the complement to one of the Uniformity curves, which he calls the Inequality curve.

When $p_i = 0.2$, $L_i = 0.047$; this means that the poorest 20 per cent of the population earns less than 5 per cent of the total income. When $p_i = 0.4$, $L_i = 0.138$; that is, the poorest 40 per cent of the population earns about 14 per cent of the total income; and so on. When $p_i = 0.8$, $L_i = 0.500$; that is the poorest 80 per cent of the population earns half of the total income. When the whole income distribution is considered, that is $p_i = 1$, then $L_i = 1$, because the whole population (the poorest 100 per cent of the population) earns all of the income. Similarly, for different values of SD: in particular, the lower the SD, the higher the Lorenz curve.

Turning to the Bonferroni index, for each p_i, the Bonferroni curve is equal to the ratio between the ordinate of the Lorenz curve L_i and p_i; for each p_i, this ratio is equal to the ratio between the mean income observed for the sub-distribution $(x_1, x_2, x_3, ..., x_i)$ with $(1,2,3,..., N)$, which we call $m(x_i)$, and the overall mean income μ. Let us consider the Bonferroni curve when SD=0.85 (Figure 2).

Figure 2: Bonferroni curves when SD varies

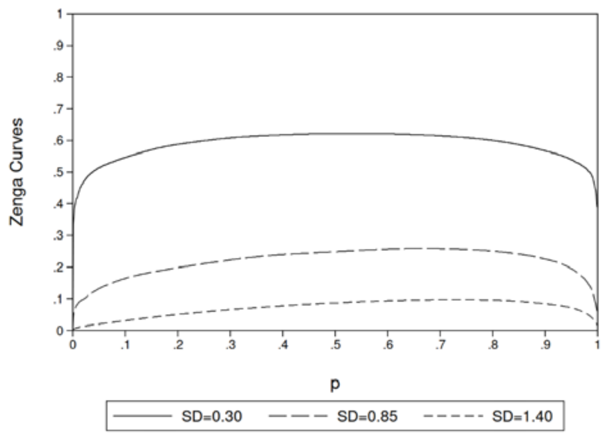


When $p_i = 0.2$, $B_i = 0.236$; this means that, for the poorest 20 per cent of the population, its mean income is about 24 per cent of the mean income evaluated for the whole population; when $p_i = 0.4$, $B_i = 0.344$, and when $p_i = 0.8$, $B_i = 0.626$. Finally, when $p_i = 1$, $B_i = 1$, as observed. In the Bonferroni case as well, the lower the SD, the higher the Bonferroni curve. Unlike the Lorenz curves (which are always convex), the Bonferroni curves are concave for low levels of p_i, and then they become convex. The inflection point depends on the SD: the lowest SD, the higher the p_i at which the inflection point occurs.

Turning to the Zenga index, for each p_i, the Zenga curve is equal to the ratio between $m(x_i)$ and $M(x_i)$. Let us consider the Zenga curve U_i when SD=0.85 (Figure 3).

When $p_i = 0.2$, $U_i = 0.198$; this means that, looking at the poorest 20 per cent of the population, their mean income is about 20 per cent of the mean income of the remaining richer individuals; when $p_i = 0.4$, $U_i = 0.239$ and when $p_i = 0.8$, $U_i = 0.250$. Finally, when $p_i = 1$, $U_i = 0.063$. In the Zenga case as well, the lower the SD, the higher the Zenga curve U_i. The Zenga U_i curves are not straight lines; however, unlike the Bonferroni curves, they show an inverse U-shape: by considering the bottom part of the income distribution – that is, moving from the lowest percentiles to the highest – $m(x_i)$ increases more than $M(x_i)$. For high levels of p_i, the relationship is reversed, because $M(x_i)$ increases more than $m(x_i)$.

Figure 3: Zenga curves when SD varies



7.3. Income weights

Turning to the representation of the three indexes as weighted sum of incomes (Equations (4), (6) and (7)), Figures 4-6 show the behaviour of the three income weights – that is $\frac{1}{N\mu}\left(1-\sum_{j=i}^{N}\frac{1}{j}\frac{\mu}{M(x_j)}\right)$ for the Zenga index, $\frac{1}{N\mu}\left(1-\sum_{j=i}^{N}\frac{1}{j}\right)$ for the Bonferroni index and $\frac{1}{N\mu}\left(2\frac{i}{N}-1-\frac{1}{N}\right)$ for the Gini index.

Whatever the inequality within the income distribution – that is, for different values of the standard deviation SD – the income weights are the same in the case of both the Gini and Bonferroni indexes, while the income weights for the Zenga index depend on the inequality of the income distribution. In particular, the Gini income weights (Figure 4) show an upward linear relationship: the weights are negative for values of p_i below the median and positive afterwards.

The Bonferroni weights show an increasing and concave relationship (Figure 5); and finally, the Zenga weights also show an increasing and concave relationship, which depends on the inequality within the income distribution (Figure 6).

Figure 4: Income weights for the Gini index

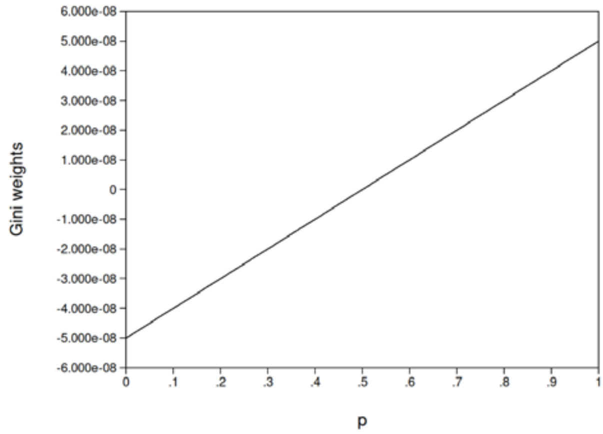


Figure 5: Income weights for the Bonferroni index

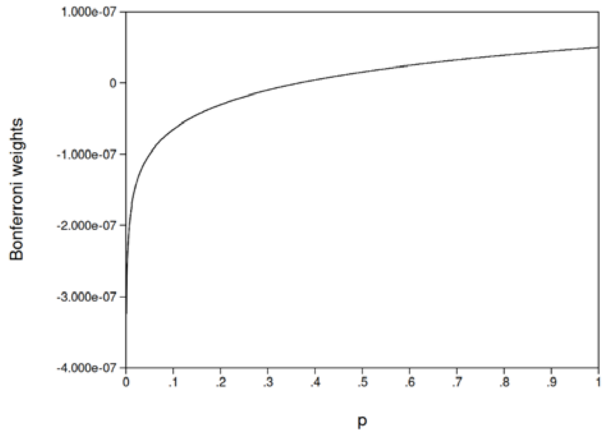
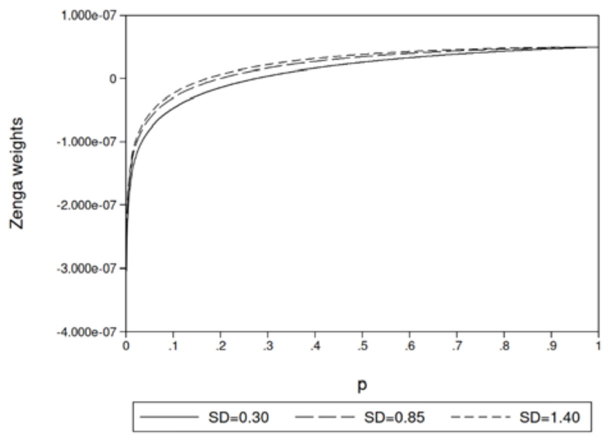


Figure 6: Income weights for the Zenga index



7.4. Derivatives

Figures 7, 8 and 9 show the behaviour of the derivatives. The Gini derivatives show an upward linear relationship, and the lower the income inequality – that is, the lower the standard deviation SD – the higher the derivatives; notably, negative derivatives occur up to the median plus $\frac{G}{2}$. The Bonferroni derivatives show an increasing and concave relationship. Negative partial derivatives occur up to approximately the $\exp\{-1+B\}^{\text{th}}$ percentile. For both the Gini and Bonferroni indexes, the derivatives never intersect (see Section 3 for the theoretical explanation of these behaviours), and the curve of the lowest inequality case dominates the corresponding curves associated with higher inequality levels.

Figure 7: Derivative of the Gini index

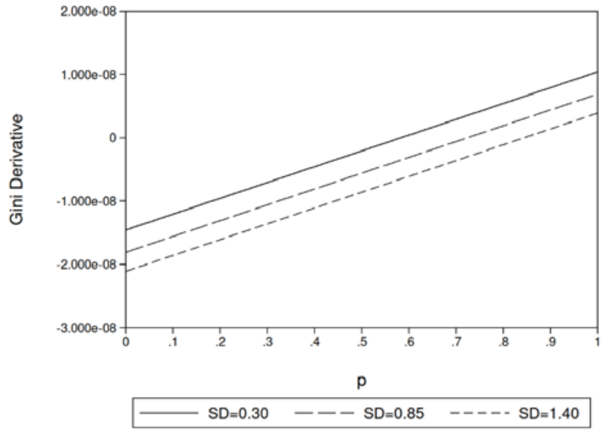
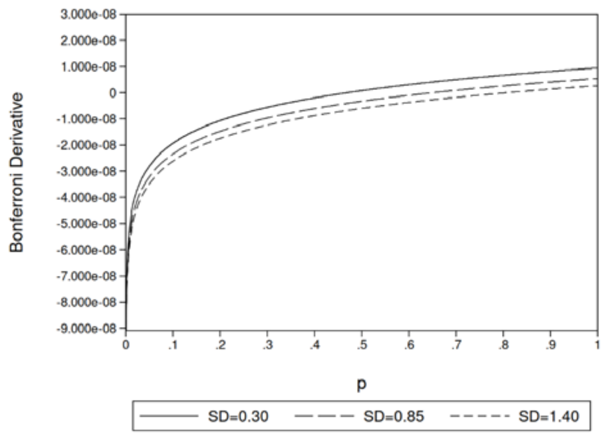
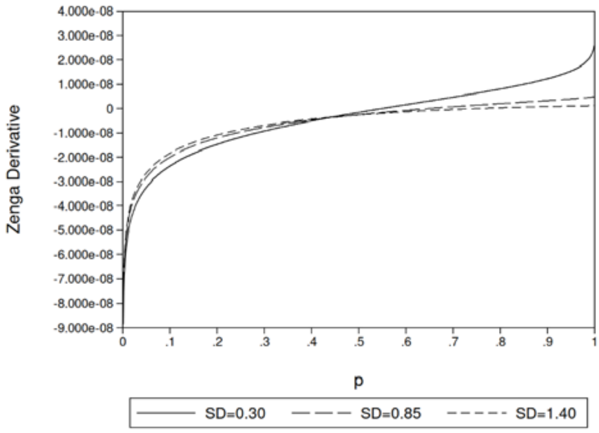


Figure 8: Derivative of the Bonferroni index



On the contrary, in the case of the Zenga index the curves do intersect. In particular, the Zenga derivatives show a concave behaviour up to a certain value of p_i, and a convex one afterwards. The higher the inequality level, the higher the curves up to approximately the median; afterwards the situation is reversed, so that the lower the inequality level, the lower the curves. Finally, our simulations show that the greater the Zenga index the more flattened towards the x-axis are the derivative lines.

Figure 9: Derivative of the Zenga index



7.5. Transfer sensitivities

By employing our 23 income distributions and having ranked the 1,000 individuals within each distribution in order from lowest to highest with respect to their incomes, in this Subsection we consider two kinds of transfers. The Kind 1 transfer considers 500 transfers taking place between two income earners, *i* and *s*, when the difference *i–s* decreases: the first re-ranking preserver transfer takes place between the richest individual (the 1,000th, the donor) and the poorest one (the first one, the recipient); the second one takes place between the second-to-last individual (the 999th, the donor) and the second poorest one (the second one, the recipient); and so on, until the transfer takes place between the 501th individual and the 500th one. The Kind 2 transfer considers instead 990 transfers taking place between two income earners, *i* and *s*, when the difference *i–s* is constant and equal to 10: the first re-ranking preserver transfer takes place between the 11th individual (the donor) and the poorest one (the first one, the recipient); the second one takes place between the 12th individual (the donor) and the second poorest one (the recipient); and so on, until the transfer takes place between the 1,000th individual and the 990th one. Figures 10, 11 and 12 show the Kind 1 transfer in the Gini, the Bonferroni and the Zenga cases, respectively; the x-axis reports the rank of the recipient. Similarly, Figures 13 and 14 show the second kind of transfer for the Gini and Bonferroni cases; finally, Figures 15A, 15B and 15C show the corresponding effect for the Zenga case. In all of these cases, the x-axis also reports the rank of the recipient.

For the Kind 1 transfer, when the difference *i–s* decreases – that is, for increasing values along the x-axis – the total differential *dG* and *dB* increases (decreases in absolute value), becoming less negative. The relationship is linear for the Gini case and concave for the Bonferroni; moreover, the relationship is the same whatever the degree of income inequality. In the Zenga index, the relationship is concave but different according to the level of income inequality: the higher the inequality of the income distribution, the higher the total differential *dZ*.

When considering Kind 2 transfer, which keeps the difference $i-s$ constant and equal to 10, *dG* does not vary, while *dB* increases (it decreases in absolute value); in both cases, the behaviour is the same for all income distributions. In the Zenga case, the behaviour still differs according to the level of inequality in the income distribution. In particular, *dZ* speedily increases (it decreases in absolute value) for ranks equal to or lower than 50 (Figure 15A). Then, when the ranks are above 50 (Figures 15B and 15C), *dZ* first continues to increase and then decreases. It starts decreasing when $i-s$ involves only the richest individuals, and the starting rank from which it decreases depends on the degree of inequality within the income distribution: 623 if SD=0.85, 844 if SD=0.85 and 988 if SD=1.40.

Figure 10: Transfer sensitivity of the Gini index – Kind 1

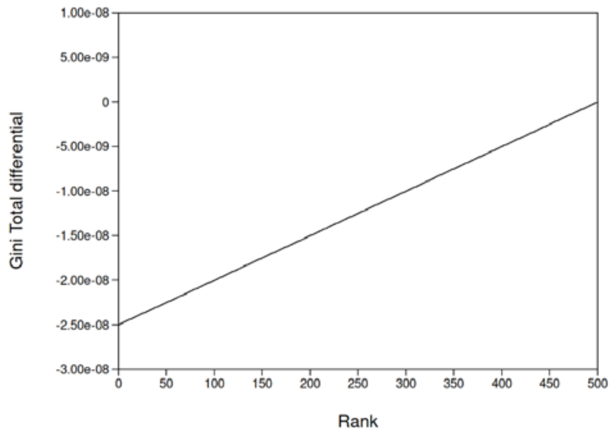


Figure 11: Transfer sensitivity of the Bonferroni index – Kind 1

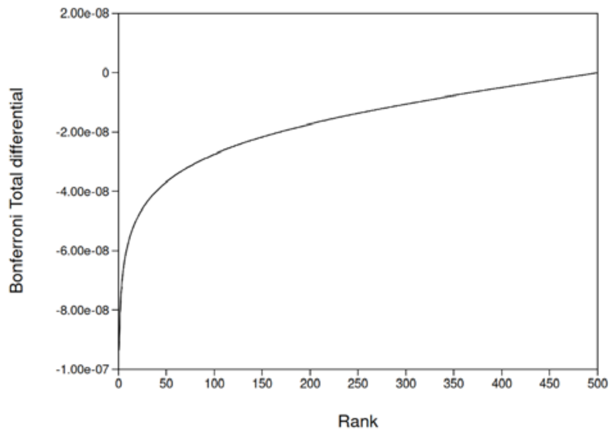


Figure 12: Transfer sensitivity of the Zenga index – Kind 1

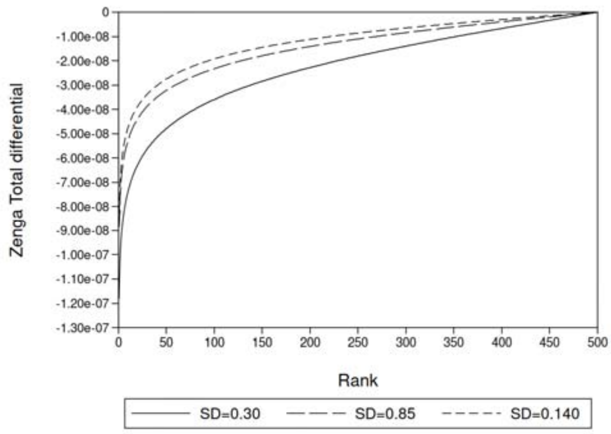


Figure 13: Transfer sensitivity of the Gini index – Kind 2

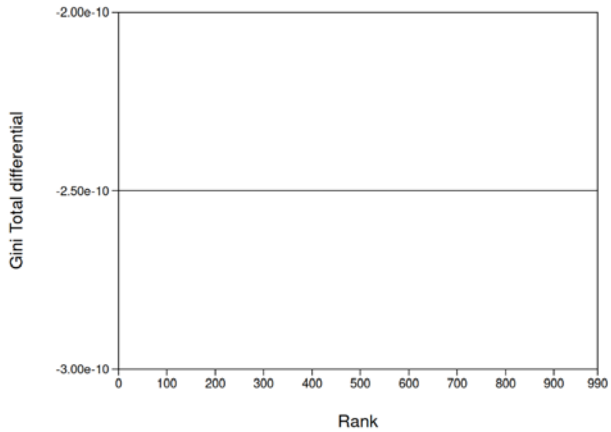


Figure 14: Transfer sensitivity of the Bonferroni index – Kind 2

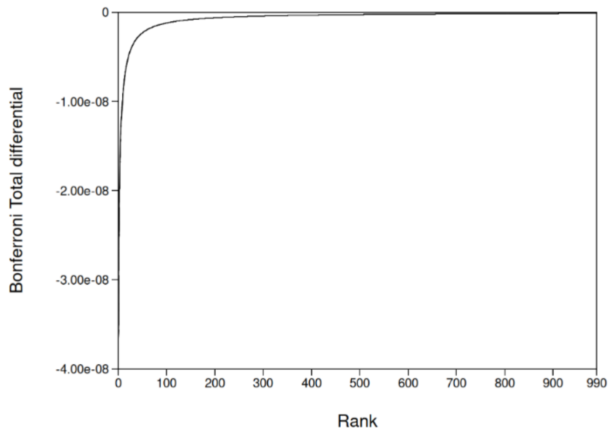


Figure 15A: Transfer sensitivity of the Zenga index – Kind 2 – Rank<50

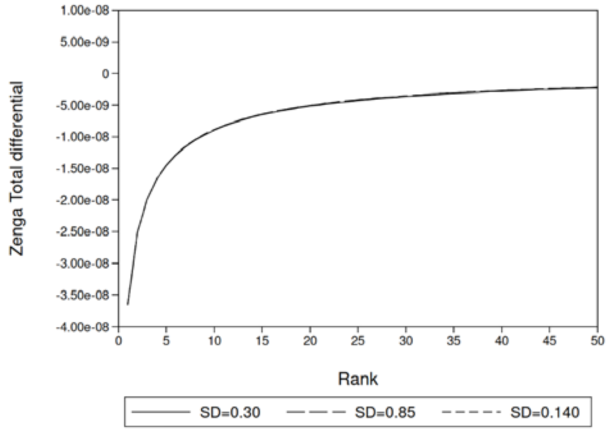


Figure 15B: Transfer sensitivity of the Zenga index – Kind 2 – 50<Rank<900

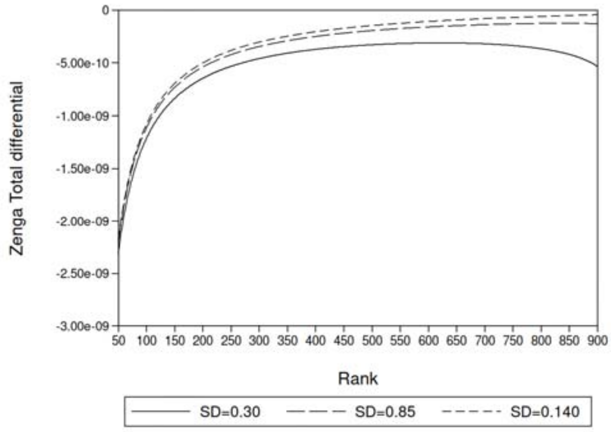
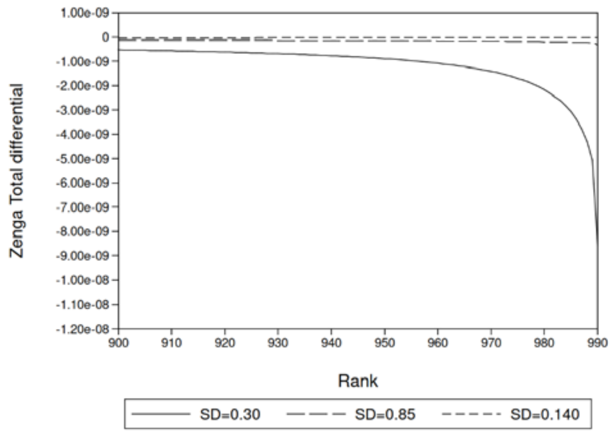


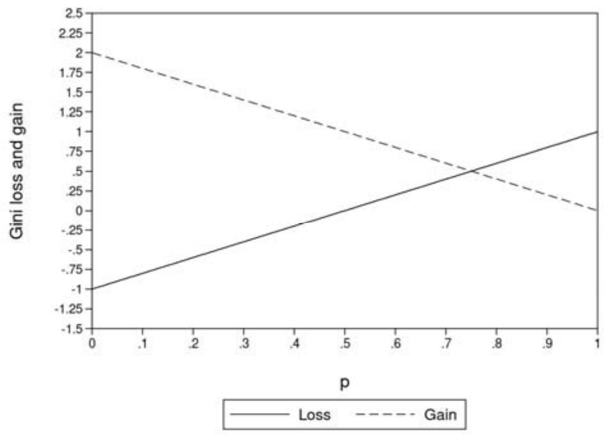
Figure 15C: Transfer sensitivity of the Zenga index – Kind 2 – Rank>900



7.6. Contribution to social welfare gain and loss

Focusing on the Gini index, Equations (16) and (17) show the weight associated with social welfare loss – that is, ${}_G\bar{w}_i\{\mathbf{x}\} = \left(2\frac{i}{N} - 1 - \frac{1}{N}\right)$ – and the corresponding social welfare gain – that is ${}_G w_i\{\mathbf{x}\} = 1 - {}_G\bar{w}_i\{\mathbf{x}\} = \left(2 - 2\frac{i}{N} + \frac{1}{N}\right)$ as discussed in Equation (15). Figure 16 shows their behaviour, which does not vary when different distributions are considered. Both the loss and the gain display a linear relationship when p_i increases: the loss increases while the gain decreases. Equation (18) – that is, ${}_G SW\{\mathbf{x}\} = \frac{1}{N}\sum_{i=1}^{N} x_i \cdot \left(2 - 2\frac{i}{N} + \frac{1}{N}\right)$ – shows how the overall level of social welfare can be evaluated; in particular, it identifies the contribution of each individual to social welfare level: $\frac{1}{N}\left(2 - 2\frac{i}{N} + \frac{1}{N}\right)x_i$. Figure 17 presents these contributions. All distributions show an inverse U-shaped form, and their maximum value differs when distributions with different degrees of inequality are considered. Focusing on the distribution with SD=0.30 – that is, the one with lower inequality – the contributions to social welfare level increase up to the 15th percentile, after which they decrease; focusing on the distribution with SD=0.85, they increase up to the 65th percentile, while they increase up to the 90th percentile when the distribution with SD=1.40 is considered.

Figure 16: Weights to determine social welfare loss and gain – Gini



Turning to the Bonferroni index, the weights that enter Equation (15) are ${}_B\bar{w}_i\{\mathbf{x}\} = 1 - \sum_{j=i}^{N}\frac{1}{j}$ and ${}_B w_i\{\mathbf{x}\} = \sum_{j=i}^{N}\frac{1}{j}$. Figure 18 illustrates their behaviour. In this case, too, they are identical for all income distributions. In particular, when p_i increases, the loss increases and it is concave downwards, while the gain decreases and it is concave upwards.

Figure 17: Contribution of each individual to social welfare – Gini

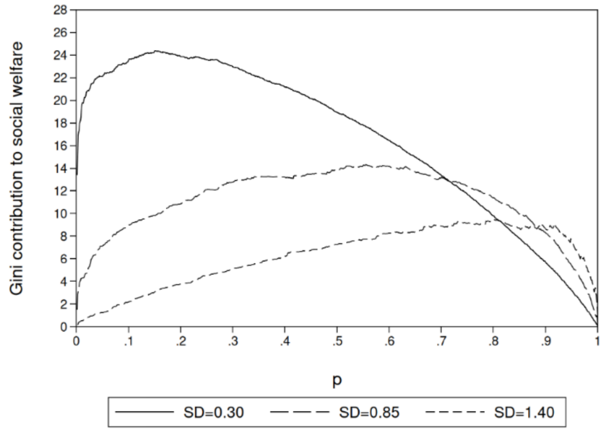
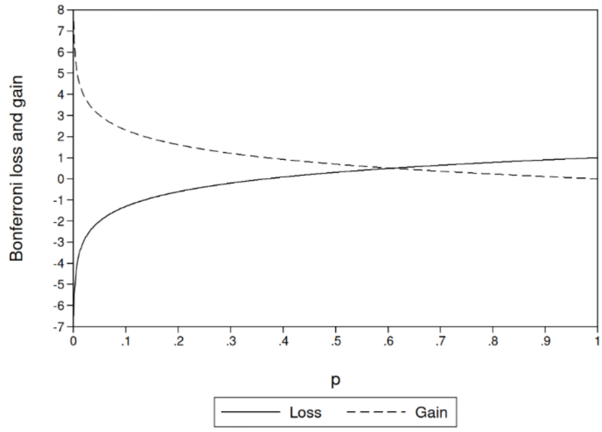


Figure 18: Weights to determine social welfare loss and gain – Bonferroni



Equation (20) – that is, ${}_{B}SW\{\mathbf{x}\} = \frac{1}{N}\sum_{i=1}^{N} x_i \cdot \sum_{j=i}^{N} \frac{1}{j}$ – identifies the contribution of each individual to the level of social welfare. Figure 19 shows these contributions; in the Bonferroni case as well, they differ according to the degree of income inequality. In particular, by focusing on the income distribution with SD=0.30, the contributions show decreasing values when p increases, and they cross the contributions of more unequal distributions; focusing on more unequal distributions, the behaviour of the contributions shows an inverse U-shaped form.

Finally, Figures 20 and 21 show the corresponding situations for the Zenga index. Here we observe different weights according to the level of inequality within our income distributions; the shapes are similar to that observed for the gain and for the loss in the Bonferroni case: when p_i increases, the Zenga losses increase and they are concave downwards; as the Zenga gains decrease, they are concave upwards. The contributions of each individual to the social welfare levels show patterns very close to those observed for the Bonferroni case.

Figure 19: Contribution of each individual to social welfare – Bonferroni

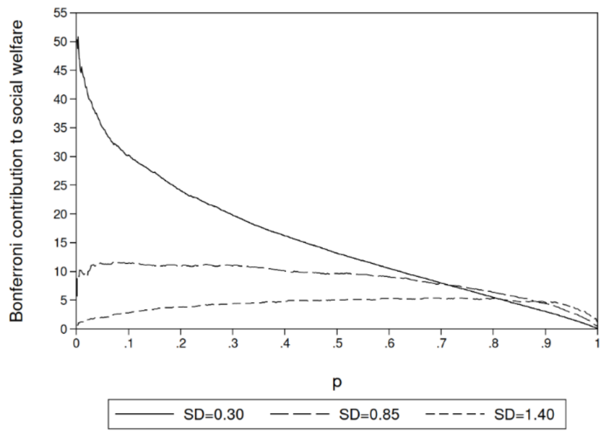


Figure 20: Weights to determine social welfare loss and gain – Zenga

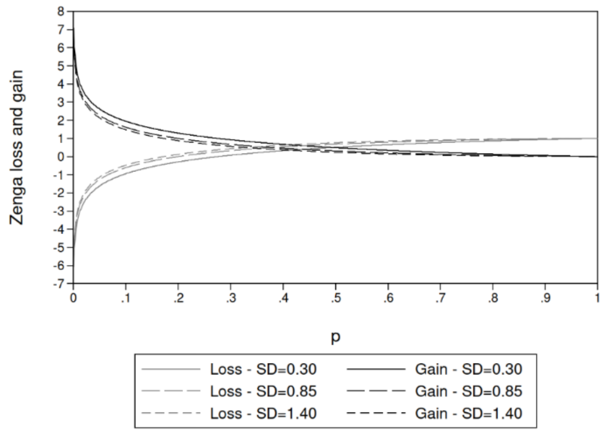
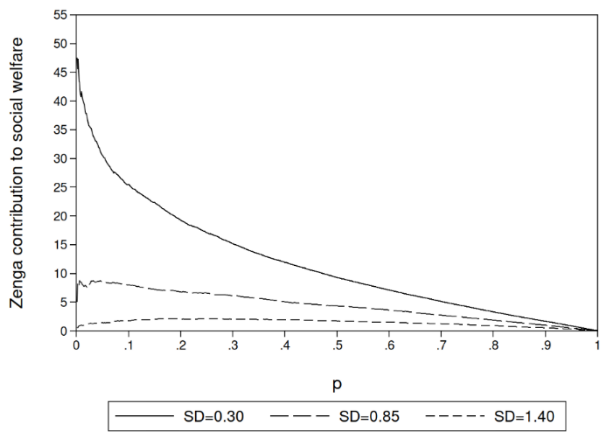


Figure 21: Contribution of each individual to social welfare – Zenga

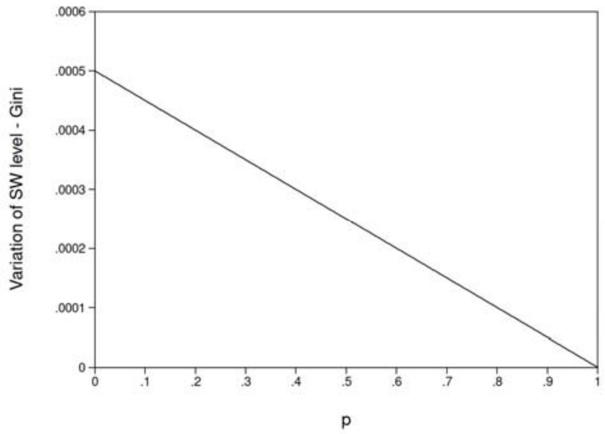


7.7. Variation in social welfare levels

An aspect that distinguishes the Zenga approach to the other two approaches is its sensitivity to measuring social welfare level variation when a change (which does not determine re-ranking) in the individual *i*'s income occurs and all other incomes remaining unchanged (see also Paragraph 3, Paragraph 7.4, and Equations (18), (20) and (22)). A small increase in the individual *i*'s income increases the overall mean income μ. What happens to the social welfare level also depends on the variation of the corresponding indexes, because ${}_{G}SW = \mu(1-G)$, ${}_{B}SW = \mu(1-B)$ and ${}_{Z}SW = \mu(1-Z)$. Equations (18), (20) and (22) focus on this aspect. By considering the effect that a change in the individual *i*'s income has on ${}_{G}SW$, ${}_{B}SW$ and ${}_{Z}SW$, the variation of the social welfare level be derived. We are then interested in evaluating $\frac{\partial_{G}SW}{\partial x_i}dx_i$, $\frac{\partial_{B}SW}{\partial x_i}dx_i$ and $\frac{\partial_{Z}SW}{\partial x_i}dx_i$. For all of our income distributions, we always consider dx_i=0.25 euro. Figure 22 focuses on the Gini index, while Figures 23 and 24 on the Bonferroni and Zenga indexes, respectively.

As it can be noted, for both the Gini and Bonferroni, a small increase in individual *i*'s income always increases the social welfare level; the effect is decreasing with respect to individual *i*'s rank in the income distribution, so that the social welfare increases more if the change in individual *i*'s income concerns poorer individuals, and it increases less if the change refers to richer individuals.

Figure 22: Variation of the social welfare level – Gini



In the Zenga index, it is not granted that the social welfare level always increases, and Figure 24 focuses on this situation. The social welfare level increases if the increase of individual *i*'s income refers to individuals up to a certain percentile. If the change occurs for the richest individuals, the level of social welfare falls. In particular, when the income inequality is low (SD=0.30), a small increase in the income of individuals ranked 781 or higher (over 1,000) reduces the social welfare level. Similarly, when SD=0.85, the social welfare level falls if the small changes refer to individuals ranked 859 or higher; finally, when SD=1.40, the social welfare level falls if the small changes refer to individuals ranked 922 or higher. Finally, we again stress that these variations are only due to the rank in the income distribution when the Gini and Bonferroni indexes are considered, whilst in the Zenga case they also depend on the initial value of the inequality: the lower the index, the more sensitive the variations.

Figure 23: Variation of the social welfare level – Bonferroni

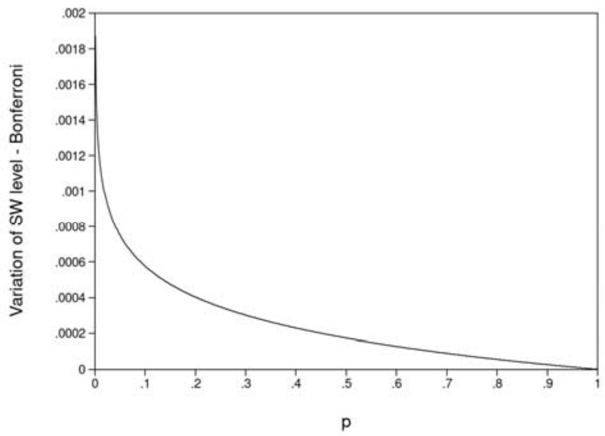
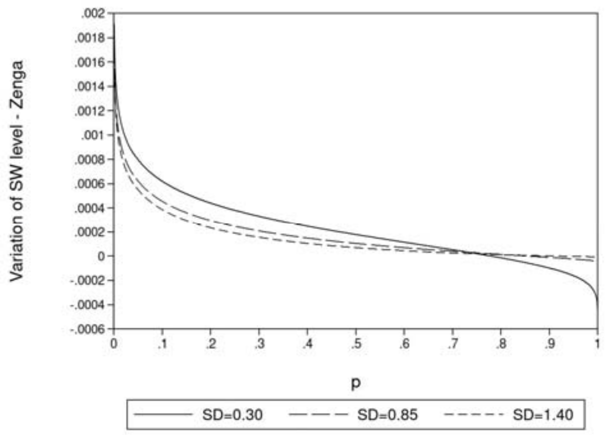


Figure 24: Variation of the social welfare level – Zenga



8. Concluding remarks

In this paper we compared Gini, Bonferroni and Zenga inequality indexes from several perspectives. We presented and discussed their main peculiarities and analysed the corresponding curves; we also represented the three indexes as a weighted sum of incomes. Derivative and total differential behaviours as well as the comparisons among the three indexes suggest investigating another aspect of the Zenga index, that is its transfer sensitivity. In this framework, we showed that the Zenga index, differently from Gini and as Bonferroni ones, satisfies the Positional Transfer Principle. Moreover, we considered the effects of distributional changes in terms of inequality indexes, with a specific focus on how the Zenga index interprets income inequality. By representing these indexes as the weighted sum of incomes as well as by considering specific kinds of income transfers involving rich and poor income units, we observed that, whatever the inequality within the income distribution, the income weights, as well as the effects of income transfers, are the same for both the Gini and the Bonferroni indexes, while they are different when the Zenga index is considered, as they depend on the inequality of the income distribution. A theoretical section focused on a short mention about deprivation: here we showed that the Zenga index rewrites as sum of two components, one of them is linked to the Bonferroni index.

We then went through the hitherto unexplored topic of the social welfare function as well as the social welfare level associated with the Zenga index; we compared them with the ones derived by employing the Gini and Bonferroni indexes. Since the social welfare level is a function of the whole income distribution for all three indexes, we found that the Gini and Bonferroni indexes always increase when the average income of the distribution increases; this is not the same in the case of the Zenga index: in this case, it is not granted that the social welfare level increases when the average income increases. When the income increase is located among the richest individuals, the social welfare can decrease when mean income increases. We can thus conclude that the Zenga inequality index reveals more than the Gini and Bonferroni indexes; in particular, it is more sensitive for evaluating income changes taking place in different parts of the income distribution.

Compliance with ethical standards

Conflict of interest. The authors declare that they have no conflict of interest.

References

Antal, E., Langel M. and Tillé Y. (2011), “Variance Estimation of Inequality Indices in Complex Sampling Designs”, Proceedings of the 58th World Statistical Congress.

Arcagni A. and Porro F. (2014), “The Graphical Representation of Inequality”, Revista Colombiana de Estadistica, 37, 419-36.

Arcagni A. (2013), “Application of Zenga’s Distribution to a Panel Survey on Household Incomes of European Member States,” Statistica & Applicazioni, 11(1), 79-102.

Atkinson A. B. (1970), “On the Measure of Inequality”, Journal of Economic Theory, 2(3), 244-263.

Bishop J., Thistle P. and Chakraborti S. (1989), Relative Deprivation and Economic Welfare: A Statistical Investigation with Gini-Based Welfare Indices, Luxembourg Income Study (LIS), Working Paper Series, Working Paper No. 32.

Blackorby C. and Donaldson D. (1978), "Measures of relative equality and their meaning in terms of social welfare", Journal of economic theory, 18(1), 59-80.

Bonferroni C. (1930), Elementi di statistica generale, Libreria Seber, Firenze.

Chakravarty, S. R. (2007). A deprivation-based axiomatic characterization of the absolute Bonferroni index of inequality, Journal of Economic Inequality, 5(3), 339-351.

Chakravarty S. R. and Muliere P. (2003), Welfare indicators: a review and new perspectives. 1. Measurement of inequality, Metron, LXI(3), 457-497.

Cowell F. A. "Gini, deprivation and complaints", in Betti G. and Lemmi A. (Eds), Advances on Income Inequality and Concentration Measures, 47-66, Routledge, 2008.

Creedy J. (2021), "Distributional comparisons using the Gini inequality measure", Chair in Public Finance, Working Papers in Public Finance, Victoria University of Wellington, 8/2021.

Ćwiek, M. and Trzcińska, K. (2023), "Assessment of goodness of fit of income distribution in France and Germany based on the Zenga distribution", Qualily & Quantity, 57(5), 4013-4027.

Dagum C. (1990), "On the relationship between income inequality measures and social welfare functions", Journal of Econometrics, 43(1-2), 91-102.

Dalton H. (1920), "The measurement of the inequality of incomes", The Economic Journal, 30(119), 348-361.

de Vergottini M. (1940), "Sul significato di alcuni indici di concentrazione", Giornale Degli Economisti e Annali Di Economia, 2(5/6), 317-347.

Gini C. (1914), "Sulla misura della concentrazione e della variabilità dei caratteri", Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti, 73, 1203-1248.

Greselin F. and Pasquazzi L. (2009), "Asymptotic Confidence Intervals for a New Inequality Measure", Communications in Statistics - Simulation and Computation, 38, 1742-56.

Greselin F., Pasquazzi L. and Zitikis R. (2010), "Zenga's New Index of Economic Inequality, Its Estimation, and an Analysis of Incomes in Italy", Journal of Probability and Statistics, Special Issue on "Actuarial and Financial Risks: Models, Statistical Inference, and Case Studies", 1-26.

Greselin F., Pasquazzi L. and Zitikis R. (2013), "Contrasting the Gini and Zenga Indices of Economic Inequality", Journal of Applied Statistics, 40, 282-97.

Greselin F., Pasquazzi L. and Zitikis R. (2014), "Heavy Tailed Capital Incomes: Zenga Index, Statistical Inference, and ECHP Data Analysis", Extremes, 17, 127-55.

Greselin F., Pellegrino S. and Vernizzi A. (2021), "The Zenga Equality Curve: A New Approach to Measuring Tax Redistribution and Progressivity", The Review of Income and Wealth, Vol. 67(4), 950-976.

Hey J. D. and Lambert P. J. (1980), Relative deprivation and the Gini coefficient: Comment, The Quarterly Journal of Economics, 95(3), 567-573.

Imedio-Olmedo L. J., Parrado-Gallardo E. M. and Bárcena-Martín, E. (2012). "Income inequality indices interpreted as measures of relative deprivation/satisfaction", Social indicators research, 109(3), 471-491.

Jedrzejczak, A. and Trzcinska K., "Application of the Zenga Distribution to the Analysis of Household Income in Poland by Socio-Economic Group," Statistica & Applicazioni, 16(2), 123-140.

Kolm, S. C. (1969), "The Optimal Production of Social Justice", in Margolis, J. and Guitton, H. (eds.), Public Economics: An Analysis of Public Production and Consumption and their Relations to the Private Sectors, Ch. 7, 145-200, London: Macmillan.

Langel M. and Tillé Y. (2012), "Inference by Linearization for Zenga's New Inequality Index: A Comparison with the Gini Index", Metrika, 75, 1093-110.

Maffenini W. and Polisicchio M. (2014), "The Potential of the I(p) Inequality Curve in the Analysis of Empirical Distributions", Statistica & Applicazioni, 12, 63-85.

Mehran F. (1976), Linear measures of income inequality, Econometrica, 44(4), 805-809.

Mussini M. (2013), "A Longitudinal Decomposition of Zenga's New Inequality Index," Statistica & Applicazioni, 11(1), 63-77.

Pasquazzi, L. and Zenga M. (2018), "Components of Gini, Bonferroni, and Zenga Inequality Indexes for EU Income Data", Journal of Official Statistics, 34, 149-80.

Pellegrino S. (2020), "The Gini coefficient: Its Origins", Department of Economics, Social Studies, Applied Mathematics and Statistics, Working Papers series, No. 70.

Pietra G. (1915), "Delle relazioni tra gli indici di variabilità", Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti, 74, 775-804.

Polisicchio M. (2008), "The Continuous Random Variable With Uniform Point Inequality Measure I(p)", Statistica & Applicazioni, 6, 137-51.

Polisicchio M. and Porro F. (2009), "A Comparison Between Lorenz L(p) Curve and Zenga I(p) Curve", Statistica Applicata, 21, 289-301.

Radaelli, P. (2008), "A Subgroups Decomposition of Zenga's Uniformity and Inequality Indexes", Statistica & Applicazioni, 6, 117-36.

Radaelli, P. (2010), "On the Decomposition by Subgroups of the Gini Index and Zenga's Uniformity and Inequality Indexes", International Statistical Review, 78, 81-101.

Runciman, W. G. (1966), Relative deprivation and social justice. A Study of Attitudes to Social Inequality in Twentieth-century England, Routledge and Kegan Paul, London.

Sen A. (1974), “Informational Bases of Alternative Welfare Approaches”, Journal of Public Economics, 3(4), 387-403.

Son, H. H. (2011), Equity and Well-Being: Measurement and Policy Practice, Routledge, London, 2011.

Yitzhaki S. (1979), Relative Deprivation and the Gini Coefficient, The Quarterly Journal of Economics, 93(2), 321-324.

Yitzhaki S. (1982), Relative deprivation and economic welfare, European Economic Review, 17(1), 99-113.

Zenga M. (2007), “Inequality Curve and Inequality Index Based on the Ratios Between Lower and Upper Arithmetic Means”, Statistica & Applicazioni, Vol. 5(1), 3-27.

Zenga, M., Radaelli P. and Zenga M. (2012), “Decomposition of the Zenga’s Inequality Index by Sources”, Statistica & Applicazioni, 9, 3-34.