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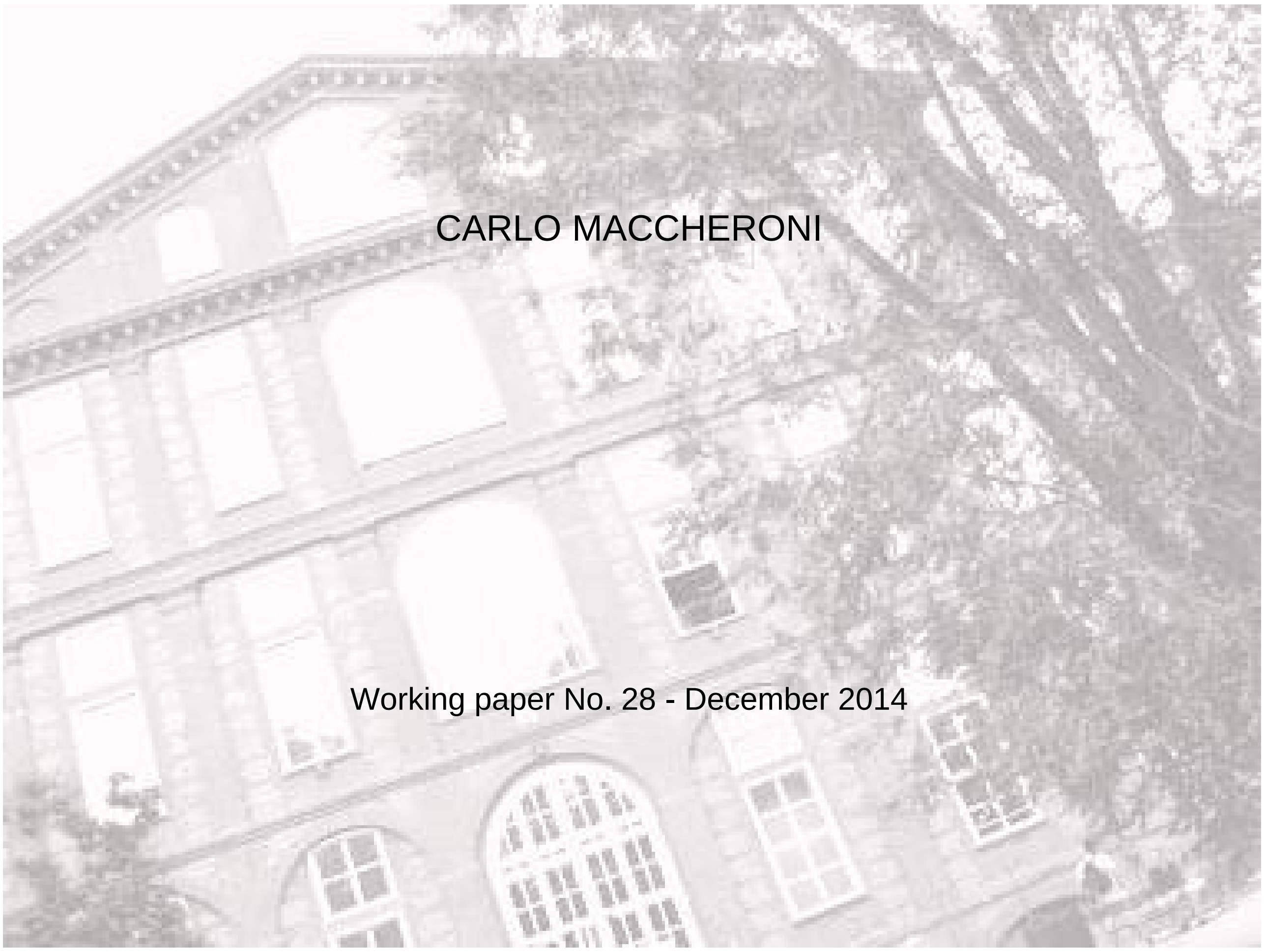
DEPARTMENT OF
ECONOMICS AND STATISTICS
WORKING PAPER SERIES

Quaderni del Dipartimento di Scienze
Economico-Sociali e Matematico-Statistiche

ISSN 2279-7114

ON THE PROBABILITY THAT NOTHING HAPPENS-II



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Working paper No. 28 - December 2014

On the Probability that Nothing Happens-II

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We analyze the probability that an isolated system (being a birth-cohort) remains unchanged along time, thus obtaining an experimental survival probability which perfectly matches with the theoretical results found in part I.

JEL Classification: C1, keyword *Probabilities.* Mathematical Subject Classification(2010): 60Axx, 60Gxx.

8.0.1 - We shall now try to apply the theory developed in part I of the previous paper, published in 2013 as n.20 of this working papers series (Rossi, 2013), to an isolated system and to the probability that it does not change.
8.0.2 - An isolated system is not only constituted by a permanent set of members, were they people, animals or objects. An isolated system should have within itself all the relationships between its members and especially all its evolution rules. Nothing must be outside it which influences it. This notion defines something which is an abstraction from reality, which is obtained by giving importance to something and disregarding something else. In particular if time is essential in the description of the system it must not be subject to limitations that come from outside but only from within.
8.0.3 - We shall study here an isolated system that depends on time, that is a birth-cohort, and examine the probability of its changing or not: in this context to change means to die and not changing means to survive. The aim should be to analyse the survival probability of an individual, but this individual cannot be a definite one, otherwise the study would be useless for everyone else. Then the individual must be identified as a generic member of a group, so that the conclusions can be useful for every member of any group that can be considered similar to the original one.

8.1 - A life table is an example of an isolated system that remains unchanged over time; indeed it provides the framework for analyzing the mortality of a closed group, the life table cohort, that shares an initial condition: the time of birth (Keyfitz, 1985). Death is the only cause of decrement in the group and death possible causes are uniformly present during the whole period. And the period ends when all the members are dead. The extreme age is not imposed but comes out of the biological evolution of the cohort and can be any time after its extinction; just for simplicity it is chosen as being one only and small enough.

8.2 - The approach to life table generally begins with the concept of the force of mortality $\mu(x)$ of the survival function $S(x)$, *i. e.* the probability for a newborn to survive to age x; $S(0)$ ($S(0) = 1$) is the initial size of the life table cohort.
The force of mortality is the instantaneous risk of dying at the exact age x,

$$\mu(x) = -\frac{1}{S(x)}\frac{d}{dx}S(x) = -\frac{d}{dx}\ln S(x) \qquad [1]$$

and it describes the proportional decline in the curve $S(x)$ at this age.

Suppose $\mu(x)$ satisfies conditions
i) for each $x \geq 0$, $\mu(x) \geq 0$.

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*Section 8.0 is actually due to Rossi.

ii) $\int_0^\infty \mu(t)\,dt = \infty$.

Since $\mu(x)$ is continuous everywhere except finitely many points, [1] can be integrated

$$S(x) = exp\left(-\int_0^x \mu(t)\,dt\right), \; x \geq 0. \qquad [2]$$

i.e.

$S(x) \geq 0, \quad S(0) = 1,$

$\lim_{x\to\infty} S(x) = 0$, and $y > x \Rightarrow S(x) > S(y)$.

then the probability that a newborn survives to *y* (given the newborn survived to *x*) is

$$p(x,y) = \frac{exp\left(-\int_0^y \mu(t)\,dt\right)}{exp\left(-\int_0^x \mu(t)\,dt\right)} \qquad [3]$$

[3] is an example according to previous conclusion 4.6; the logarithmic transformation of [3] in fact gives an issue of a kind

$$G(x,y) = g(y) - g(x) \qquad [4]$$

9.0 - As an application of the approach developed in the previous pages we may use the recent reconstruction of the 1948 cohort life table (Maccheroni, 2011).
Such a work proved necessary because no official cohort mortality tables for the overall population exist in Italy. There are two cohort tables (called RG48 and IP55, pertaining to the 1948 and 1955 cohorts) used by Italian insurance companies to determine premiums. These life tables were made by ANIA, which used the demographic forecast of the Ragioneria Generale dello Stato (RGS,1995) regarding RG48 (1998); however the mortality decline happened to be greater than predicted.

9.0.1 - Here we are going to present the main results of a new 1948 cohort life table (fig. 1), which was provided in a research that proposed some benchmarking in the construction of demographics to compute premiums (Maccheroni, 2011). The cohort life table has been constructed on the Istat period life table available during the years 1974-2008 (Istat, 2001) as regards the age group 26-60. The other two age groups (0-25 and 61-102) have been provided respectively by a mortality reconstruction which covers the period 1948-1973 (Maccheroni and Locatelli, 1999) and by a mortality forecast up to 2050 (Maccheroni, 2011).

9.0.2 - This forecast is carried out with the time series method of Lee and Carter (1992); the forecast model has the form:

$$m_{xt} = exp(a_x + b_x k_t + \varepsilon_{xt}) \qquad [5]$$

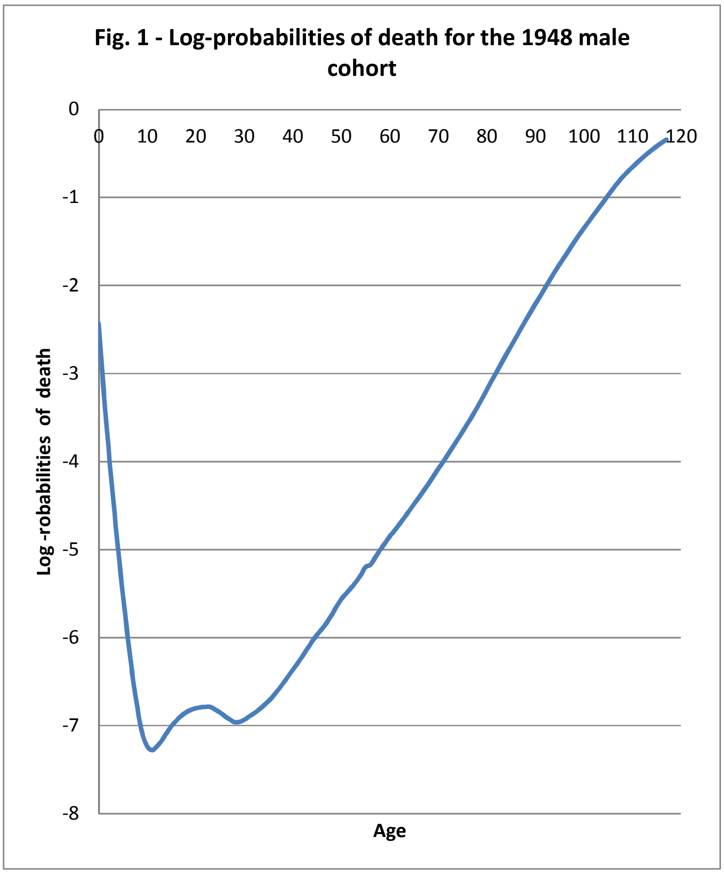
with m_{xt} the observed age-specific death rate (number of deaths during age interval ($x, x+1$) per person-year at the same age) over time t ($x = 0, 1, \ldots, 102; t = 1974, 1975, \ldots, 2008$) and the parameters:
a_x the standard age-pattern of mortality; a_x is the average of the logarithm of m_{xt},
b_x the age-specific pattern of mortality change,
k_t the period effect,
ε_{xt} a set of random disturbances.
The parameters b_x and k_t can be estimated via singular value decomposition (SVD) of the matrix $\{\ln m_{x\,t} - a_x\}$; the first left and right singular vectors give initial estimates of b_x and k_t respectively. To provide a unique solution for b_x and k_t two normalizations are required

$$\sum_x b_x = 1 \quad \text{and} \quad \sum_t k_t = 0$$

The period effect k_t has been modelled as a random-walk with drift (Lee and Carter, 1992) and the death rates forecasts are made using [5] after computing the future values for k_t (t= 2009,..., 2050).
Death rates m_x are then usually transformed into probabilities that a person exact age x will die within one year q_x, (Wunsch and Termote, 1978) by the relation,

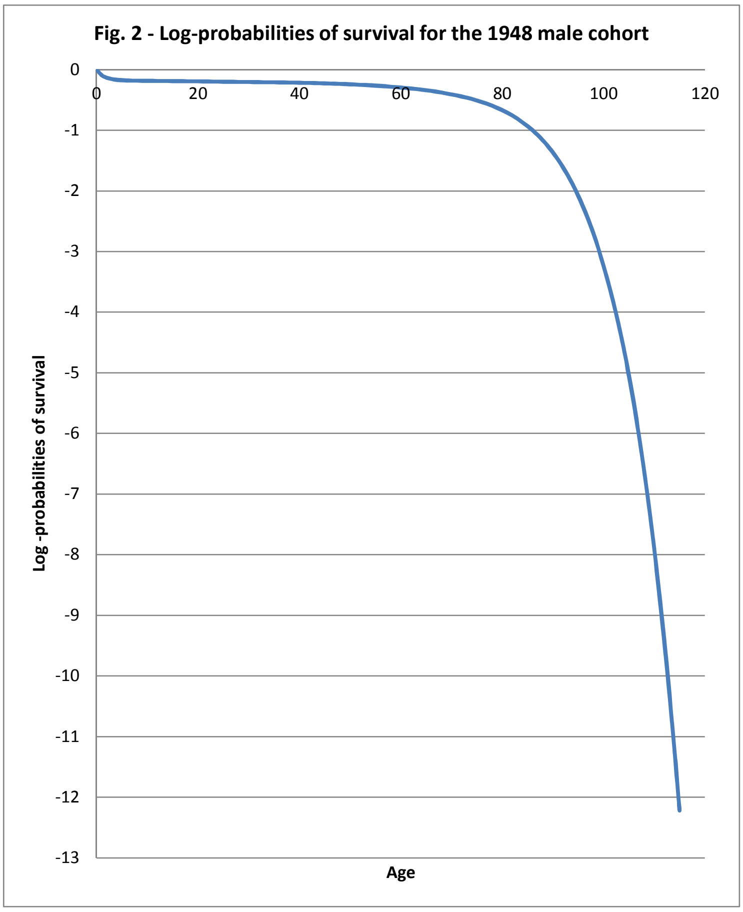
$$q_x = \frac{2\, m_x}{2 + m_x} \qquad [6]$$

q_x are the basis of the life table and they represent the probability that a person of exact age x will die before attaining exact age $x+1$.



9.1 - We thus construct a matrix ($x = 0, \ldots 102; t = 1948, \ldots 2050$) of probability of death and the secondary diagonal of this matrix describes therefore the mortality pattern of the 1948 cohort up to age 102; finally the method of Denuit and Goderniaux (2005) has been used in estimating q_x at advanced ages (*103 and over*).

Figure 1 exhibits the probabilities of death for the 1948 male cohort ($q_{x,1948}$) which are about the central (median) scenario (the more probable one) given by the model [5]. Mortality in the first years of life is typically much higher than in the immediately following years and fig. 1 shows that the abscissa of the first $q_{x,1948}$ minimum (fig. 1), which is also the global one, is at age 12. The abscissa of the other minimum is 27 years and between ages 12 and 27, where mortality is growing, there is the so-called accident hump, mostly due to road accident traumas. After age 27, $q_{x,1948}$ increases progressively.



9.2 - The usual complete life table is computed from the sequence of q_x and an additional function of the life table is the probability of survival $p_x = 1 - q_x$, which represents the probability that an individual exactly x years of age will survive to exact age $x+1$; thus we can also arrive at $S(x)$ [2] as follows

$$S(x) = \prod_{t=0}^{x-1} p_t \qquad [7]$$

for any integer x.

It is our purpose to compute and illustrate (Appendix, tab. 1) the $p(x_c,y)$ probabilities respecting the male 1948 cohort, *i.e.* the probability that a newborn in c ($c=1948$), who survived to the exact age x ($x= 0, 1, \ldots 114$), will still survive exact age y ($y = x+t;\ t = 1, 2, \ldots;\ p(x_c,x) = 1;\ p(x_c,115) = 0$); figure 3 exhibits graphically, age by age, these probabilities.

A life table is an example of a (finite) Markov chain (Chiang, 1968) and $p(x_c,y)$ may be expressed as the product of two separate probabilities

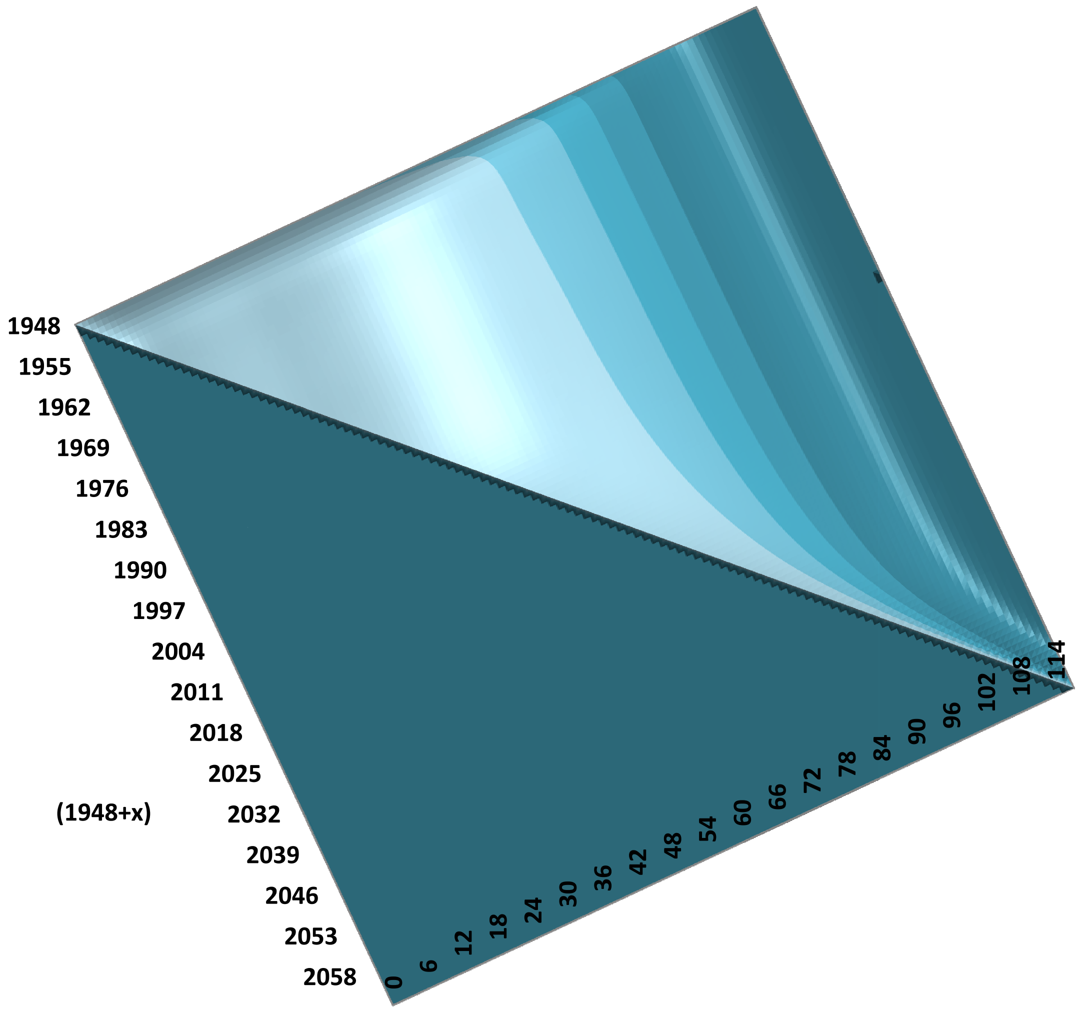
$$p(x_c,y) = p(x_c,z)\, p(z_c,y)$$

($x_c < z_c < y$). A numerical example would be $p(5_{1948},103) = p(5_{1948},9)\, p(9_{1948},103)$ and the values in this case are (Appendix, tab.1):

$$0{,}01753 = 0{,}99087 \times 0{,}0177$$

9.3 - Theoretical and practical survival modelling in demography perfectly agree with the theoretical approach developed in previous sections, so that both reinforce each other.

Fig. 3 - Probability that a live birth in 1948 surviving to the exact age x – which happens in (1948+x) – will survive up to exact age y



From [5] we have

$lg\ p(x_c,y) = g(y) - g(x_c)$

and from the requirements for a survival function we can see that for $x_{1948} = 0$, $lg\ p(x_{1948},y) = lg\ S(y)$, *i. e.* the logarithm of the survival function for the 1948 male cohort and it is the function $g(t)$, as found in part I, section 4. The results of these calculations are shown in fig. 2 and are obtained by the data in the first line of the matrix in Appendix, tab. 1.

9.4 - All the $p(x_c,y)$ results are shown in figure 3[1], where the profile seems to rise in the very first years of life because this cohort has experienced an elevated child mortality; later on the $p(x_c,y)$ shape declines progressively, particularly at old ages, when the survival curves show a rapid decline (fig. 3) and also the longevity pattern of persons at the advanced ages drops too.

These survival curves could be described by the Weibull type survival function (C. Maccheroni, 1998, Dugan *et al*.2005) which can be expressed as

$$S(y) = exp\{-[(y - x_c)/m]^b\} \qquad (y \geq x_c;\ x_c \geq 0,\ b, m > 0) \qquad [8]$$

where x_c is the "minimum life", m is a scale parameter and b is a shape parameter. As the relationship [8] does not perform well at younger ages, the curve [8] has been fitted to 1948 cohort survival experience (tab. 1) only at old ages ($x_c \geq 60$) where the curve fitting provides a valuable model.
On the diagonal of the plane (x,y) (fig. 3) there is the fixed sequence of minimum life x_c, ($x_c \geq 60$), while parameters b and m are to be estimated: this can be done by the least squares method. The b and m estimates at each integer x_c value in the age group *60-85* have been evaluated this way.
The lines of $b(x_c)$ (fig. 4) and $m(x_c)$ (fig. 5) plotted show quadratic relations that can be obtained by regression analysis as follows:

$$b(x_c) = - 0,0002\ x_c^{\ 2} + 0,0178\ x_c + 1,8491 \qquad (R^2 = 0,9999)$$

$$m(x_c) = 0,0073\ x_c^{\ 2} - 1,7339\ x_c + 103,12 \qquad (R^2 = 1)$$

R^2 values $(0 \leq R^2 \leq 1)$ provide results of the goodness-of-fit analyses.
So the $p(x_c,y)$ survival function in the age group 60-85 can be determined by the following equation

$$p(x_c,y) = exp\{-[(y - x_c)/ m(x_c)\ ]^{\ b(x_c)}\} \qquad [9]$$

$x_c = 60, 61, \ldots, 85;\ y > x_c$.

[1] In fig. 3 the values of *y*- axis run parallel to the axis itself, so as not to interfere with the $p(x_c,y)$ curve.

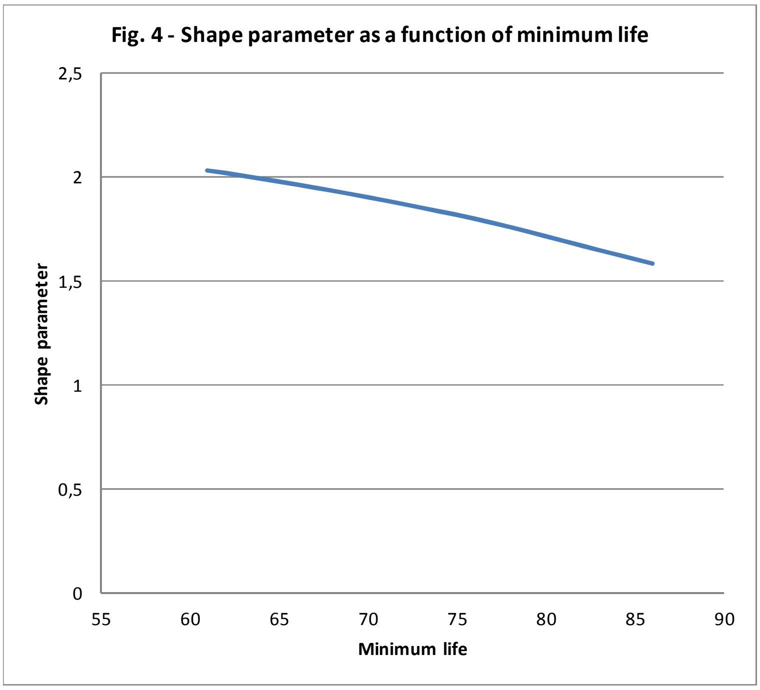


Fig. 4 - Shape parameter as a function of minimum life
Shape parameter
2,5
2
1,5
1
0,5
0
55
60
65
70
75
80
85
90
Minimum life

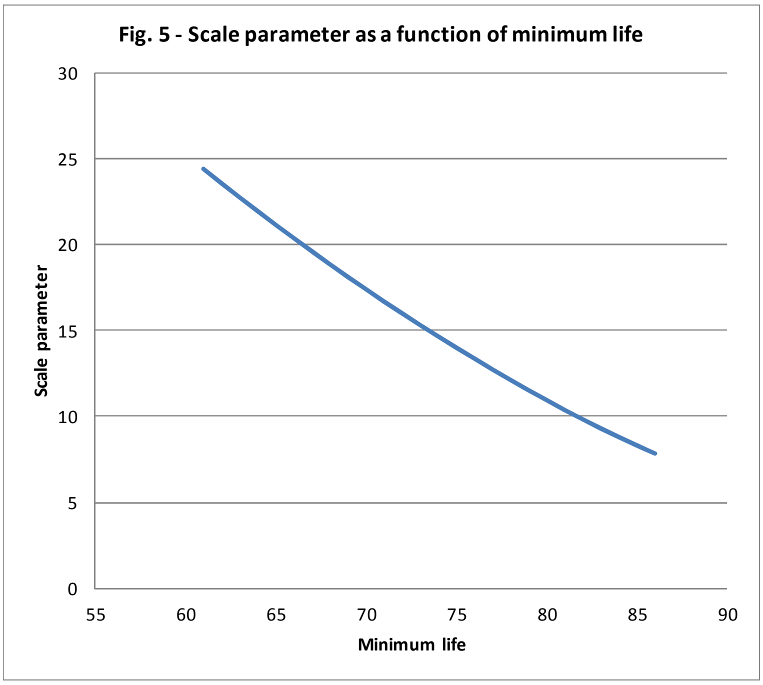


Fig. 5 - Scale parameter as a function of minimum life
Scale parameter
30
25
20
15
10
5
0
55
60
65
70
75
80
85
90
Minimum life

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	0	1	2	3	4	5	6	7	8	9	10	...	101	102	103	104	105	106	107	108	109	110	111
0	1	0,9122	0,8743	0,8555	0,8454	0,8397	0,8364	0,8343	0,8330	0,8321	0,8314	...	0,0293	0,0211	0,0147	0,0099	0,0065	0,0040	0,0024	0,0014	0,0007	0,0004	0,0002
1		1	0,9584	0,9378	0,9268	0,9206	0,9169	0,9146	0,9132	0,9122	0,9114	...	0,0321	0,0231	0,0161	0,0109	0,0071	0,0044	0,0026	0,0015	0,0008	0,0004	0,0002
2			1	0,9785	0,9670	0,9605	0,9567	0,9543	0,9528	0,9517	0,9510	...	0,0335	0,0241	0,0168	0,0114	0,0074	0,0046	0,0027	0,0015	0,0008	0,0004	0,0002
3				1	0,9882	0,9816	0,9777	0,9752	0,9737	0,9726	0,9718	...	0,0343	0,0247	0,0172	0,0116	0,0076	0,0047	0,0028	0,0016	0,0008	0,0004	0,0002
4					1	0,9933	0,9893	0,9869	0,9853	0,9842	0,9834	...	0,0347	0,0249	0,0174	0,0118	0,0076	0,0048	0,0028	0,0016	0,0009	0,0004	0,0002
5						1	0,9960	0,9935	0,9920	0,9909	0,9900	...	0,0349	0,0251	0,0175	0,0118	0,0077	0,0048	0,0028	0,0016	0,0009	0,0004	0,0002
6							1	0,9975	0,9959	0,9948	0,9940	...	0,0350	0,0252	0,0176	0,0119	0,0077	0,0048	0,0029	0,0016	0,0009	0,0004	0,0002
7								1	0,9984	0,9973	0,9965	...	0,0351	0,0253	0,0176	0,0119	0,0078	0,0048	0,0029	0,0016	0,0009	0,0004	0,0002
8									1	0,9989	0,9981	...	0,0352	0,0253	0,0177	0,0119	0,0078	0,0048	0,0029	0,0016	0,0009	0,0004	0,0002
9										1	0,9992	...	0,0352	0,0253	0,0177	0,0120	0,0078	0,0048	0,0029	0,0016	0,0009	0,0004	0,0002
10											1	...	0,0353	0,0254	0,0177	0,0120	0,0078	0,0048	0,0029	0,0016	0,0009	0,0004	0,0002
												...	...	...	...	...	...	...	...	...	...	...	...
100													0,7402	0,5327	0,3719	0,2512	0,1633	0,1018	0,0604	0,0341	0,0182	0,0093	0,0044
101													1	0,7197	0,5024	0,3393	0,2207	0,1375	0,0816	0,0461	0,0246	0,0125	0,0060
102														1	0,6981	0,4715	0,3066	0,1910	0,1135	0,0640	0,0343	0,0174	0,0083
103															1	0,6753	0,4392	0,2736	0,1625	0,0917	0,0491	0,0249	0,0120
104																1	0,6503	0,4051	0,2406	0,1358	0,0726	0,0369	0,0177
105																	1	0,6230	0,3700	0,2088	0,1117	0,0567	0,0272
106																		1	0,5940	0,3352	0,1793	0,0910	0,0437
107																			1	0,5643	0,3019	0,1532	0,0736
108																				1	0,5350	0,2715	0,1304
109																					1	0,5074	0,2437
110																						1	0,4803
111																							1

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