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EQUILIBRIUM SELECTION THROUGH p_u-DOMINANCE



Equilibrium selection through \mathbf{p}_u -dominance

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Abstract

This paper introduces and discusses the concept of \mathbf{p}_u -dominance in the context of finite games in normal form. It then presents the \mathbf{p}_u -dominance criterion for equilibrium selection, a generalization of the risk-dominance criterion to games with more than two players.

Keywords: equilibrium selection; normal form games; \mathbf{p}_u -dominance. JEL Classification: C72, C73.

1 Introduction

Multiplicity of equilibria is a feature that characterizes many strategic interactions. A typical example is a coordination game where multiple pure strategy Nash equilibria occur when players' actions match. Whenever multiple equilibria exist, equilibrium selection obviously becomes a key issue, both from a normative and a positive point of view (see for instance Kim, 1993, and Haruvy and Stahl, 2007).

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In this paper, we introduce the concept of \mathbf{p}_u -dominance and we use it as a criterion for equilibrium selection in the context of finite one-shot simultaneous games. In a nutshell, the \mathbf{p}_u -dominance criterion selects the equilibrium whose supporting actions turn out to be \mathbf{p}_u -dominant for the less stringent set of beliefs. More precisely, we say that an equilibrium is \mathbf{p}_u -dominant for the vector $\mathbf{p}_u = (p_1, ..., p_n)$ if, for every player $i \in \{1, ..., n\}$, *i*'s equilibrium action best responds to any conjecture according to which any player $j \neq i$ plays his equilibrium action with a probability of at least p_i and uniformly randomizes the remaining probability over his alternative actions.¹ \mathbf{p}_u -Dominance thus mimics the mental process according to which every agent evaluates the likelihood of an equilibrium by focusing on the probability of the actions that sustain it, while assuming a simplified uniform distribution of the other players' alternative actions. The approach is therefore similar to the one that seems to drive the actual behavior of the majority of players in normal form games: choose the strategy that best responds to the belief that the opponents uniformly randomize over their action space (see Costa-Gomes and Weizsäcker, 2008, for experimental evidence on this claim).

An important feature of the \mathbf{p}_u -dominance criterion is thus the fact that the criterion evaluates the actions that support an equilibrium in light of a conjecture (i.e., a probability distribution) that assigns a positive probability to the event that (some of) the opponents deviate and do not play their corresponding action. As such, \mathbf{p}_u -dominance considers not only the profitability of playing a certain equilibrium action but also its riskiness.

The \mathbf{p}_u -dominance criterion thus tackles the issue of equilibrium selection using the same intuition that underlies other concepts such as risk-dominance (Harsanyi and Selten, 1988)

¹The fact that the conjecture postulates that players use a uniform distribution to randomize over the actions that do not belong to the profile under scrutiny explains the subscript "u" in the term " \mathbf{p}_u -dominance". This should not be confused with the concept of u-dominance (Kojima, 2006) that has been proposed as a criterion for equilibrium selection based on perfect foresight dynamics and that assumes that the number of opponents adopting a certain strategy follows a uniform distribution.

or **p**-dominance (Morris et al., 1995, Kajii and Morris, 1997): if multiple equilibria exist and agents do not know which equilibrium will arise, they will coordinate their expectations on the one that better solves the trade-off between risk and return. In the second part of the paper, we more thoroughly investigate the relationships that exist between \mathbf{p}_u -dominance, risk-dominance, and **p**-dominance. We show that the \mathbf{p}_u -dominance criterion and the riskdominance criterion coincide in any 2x2 games, but \mathbf{p}_u -dominance is a more general concept, as it can also be applied to games that have more than two players. On the other hand, \mathbf{p}_u -dominance turns out to be a special case of **p**-dominance, but it has the advantage of being more easily computable and it also more closely mimics the heuristic approach that the majority of individuals actually use.

2 The concept of p_u -dominance

Consider a generic one shot simultaneous game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where $N = \{2, ..., n\}$ denotes the set of players, $A_i = \{a_1, ..., a_{k_i}\}$ with $k_i \geq 2$ is the action space of player $i \in N$ (notice that its cardinality $k_i = |A_i|$ may differ across players), and $u_i : A \to \mathbb{R}$ is the payoff function with $A = \times_{j \in N} A_j$. As usual, a profile $a^* \in A$ is a Nash equilibrium of G if the relation $u_i(a_i^*, a_{-i}^*) \geq (a_i, a_{-i}^*)$ holds for every $i \in N$ and every $a_i \in A_i$.

Now take any action profile \hat{a} (not necessarily a Nash equilibrium) and define the vector $\mathbf{p}_u(\hat{a}) = (p_1(\hat{a}), ..., p_n(\hat{a}))$. The vector $\mathbf{p}_u(\hat{a})$ assigns to every player $i \in N$ a specific hypothetical conjecture about what every other agent will play (i.e., a probability distribution defined on every A_j for $j \neq i$). More precisely, $\mathbf{p}_u(\hat{a})$ postulates that each agent $i \in N$ believes that any other player $j \in N_{-i}$ will play action \hat{a}_j with probability $p_i(\hat{a})$ and any other action $a_j \neq \hat{a}_j$ with probability $\frac{(1-p_i(\hat{a}))}{k_i-1}$.

We then say that the action profile \hat{a} is \mathbf{p}_u -dominant for $\mathbf{p}_u(\hat{a}) = (p_1(\hat{a}), ..., p_n(\hat{a}))$ if, for any player $i \in N$ and any $j \in N_{-i}$, action \hat{a}_i is a best response to any probability distribution $\lambda \in \Delta(A_{-i})$ that assigns at least probability $p_i(\hat{a})$ to the event of j playing action \hat{a}_j and lets j uniformly randomize with the remaining probability over his alternative actions.

Definition 1 Action profile $\hat{a} \in A$ is \mathbf{p}_u -dominant with $\mathbf{p}_u(\hat{a}) = (p_1(\hat{a}), ..., p_n(\hat{a}))$ if for all $i \in N$, $a_i \neq \hat{a}_i$ and all $\lambda \in \Delta(A_{-i})$ with $\lambda(\hat{a}_j) \ge p_i(\hat{a})$ and $\lambda(a_j) = \frac{(1-\lambda(\hat{a}_j))}{k_j-1}$ for all $a_j \neq \hat{a}_j$ and $j \in N_{-i}$,

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) u_i(\hat{a}_i, a_{-i}) \ge \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) u_i(a_i, a_{-i}).$$
(1)

Some basic concepts in game theory can be formulated in terms of \mathbf{p}_u -dominance. For instance, an equilibrium in dominant strategies is \mathbf{p}_u -dominant with $\mathbf{p}_u = (0, ..., 0)$. In contrast, there is no vector \mathbf{p}_u for which a profile that contains dominated actions can turn out to be \mathbf{p}_u -dominant.

Focusing on Nash equilibria in pure strategies, every equilibrium is \mathbf{p}_u -dominant with $\mathbf{p}_u = (1, ..., 1)$: notice in fact that with such a (degenerate) vector of probability distributions, the definition of \mathbf{p}_u -dominance boils down to the requirement that every action in the profile is a best response to the actions taken by the other players, i.e., the definition of a Nash equilibrium. But notice also that in general an equilibrium a^* is \mathbf{p}_u -dominant also for some other vectors $\mathbf{p}_u \leq (1, ..., 1)$. More precisely, if the equilibrium a^* is \mathbf{p}_u -dominant with $\mathbf{p}_u(a^*)$ then a^* is also \mathbf{p}_u -dominant for any $\mathbf{p}'_u(a^*) \geq \mathbf{p}_u(a^*)$.

If multiple equilibria exist (say the set of equilibria is given by $A^* = \{a^{*1}, ..., a^{*\Psi}\}$ with $\Psi \geq 2$), what characterizes a specific equilibrium $a^{*\psi} \in A^*$ is the smallest $\mathbf{p}_u(a^{*\psi})$ for which $a^{*\psi}$ turns out to be \mathbf{p}_u -dominant. This vector, which we indicate with $\mathbf{\bar{p}}_u(a^{*\psi}) = (\bar{p}_1(a^{*\psi}), ..., \bar{p}_n(a^{*\psi}))$, reports the smallest probabilities $\bar{p}_i(a^{*\psi})$ for which $a_i^{*\psi}$ is a best response to the associated conjecture. As such, $\bar{p}_i(a^{*\psi})$ provides a measure of the riskiness of playing the equilibrium action $a_i^{*\psi}$ as well as a tool to identify the equilibrium upon which players' expectations should coordinate. In particular, and in the same spirit of **p**-dominance

(Kajii and Morris, 1997), the \mathbf{p}_u -dominance criterion selects the equilibrium (or the subset of equilibria) $a^{*\psi} \in A^*$ for which the following relation holds:

$$\sum_{i=1}^{n} \bar{p}_i(a^{*\psi}) \le \sum_{i=1}^{n} \bar{p}_i(a^{*\xi}) \text{ for any } a^{*\xi} \in A^*$$
(2)

In other words, the \mathbf{p}_u -dominance criterion selects the equilibrium whose supporting actions emerge as \mathbf{p}_u -dominant under the less stringent set of beliefs. Notice that if the game is symmetric requirement (2) holds if and only if $\mathbf{\bar{p}}_u(a^{*\psi}) \leq \mathbf{\bar{p}}_u(a^{*\xi})$, i.e., $a^{*\psi}$ is the equilibrium characterized by the smallest $\mathbf{\bar{p}}_u$ vector.

As an example, consider the following 3x3 coordination game where the action space of player $i \in \{A, B, C\}$ is given by $A_i = \{H, M, L\}$. Player A chooses the row, player B chooses the column, and player C chooses the matrix. In each cell payoffs appear in the order u_A, u_B, u_C .

	$a_C = H$		
	$a_B = H$	$a_B = M$	$a_B = L$
$a_A = H$	3,3,3	2, 0, 2	0, 1, 2
$a_A = M$	0, 2, 2	0, 0, 2	2, 3, 1
$a_A = L$	1, 2, 0	3, 2, 1	1, 1, 1

		$a_C = M$	
	$a_B = H$	$a_B = M$	$a_B = L$
$a_A = H$	2, 2, 0	2, 0, 0	1, 3, 2
$a_A = M$	0, 2, 0	3, 3, 3	2, 1, 2
$a_A = L$	3, 1, 2	1, 2, 2	0, 0, 1

a_C	=L
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e			
	$a_B = H$	$a_B = M$	$a_B = L$
$a_A = H$	2, 0, 1	1, 2, 3	1, 1, 1
$a_A = M$	2, 1, 3	2, 2, 1	1, 0, 0
$a_A = L$	1, 1, 1	0, 1, 0	3, 3, 3

The game has three Nash equilibria: $a^{*1} = (H, H, H)$, $a^{*2} = (M, M, M)$, and $a^{*3} = (L, L, L)$. These equilibria are Pareto equivalent such that the Pareto dominance criterion

(i.e., the criterion that selects the Pareto superior equilibrium with the argument that this is the outcome upon which agents' expectations should converge) does not refine the set A^* .

The \mathbf{p}_u -dominance criterion selects instead a specific equilibrium. In order to find it, one needs to compute for every equilibrium the functions $E_{p_i}(a_i)$, i.e., the expected payoff of action $a_i \in A_i$ under player *i*'s conjecture that each opponent plays action a_j^* with probability p_i and each of his alternative actions with probability $\frac{1-p_i}{2}$. Then, by imposing the conditions $E_{p_i}(a_i^*) \ge E_{p_i}(a_i)$ for any $a_i \ne a_i^*$, we find the components of the vector $\bar{\mathbf{p}}_u$, i.e., the smallest vector for which the equilibrium $a^{*\psi}$ is \mathbf{p}_u -dominant. Finally, we apply the \mathbf{p}_u -dominance criterion and select the equilibrium characterized by the smallest $\bar{\mathbf{p}}_u$.

For instance, starting from the equilibrium a^{*1} and focusing without loss of generality on player A (the game is symmetric), we have the following: $E_{p_i}(H) = \frac{5}{4}p_i^2 + \frac{1}{2}p_i + \frac{5}{4}$, $E_{p_i}(M) = -2p_i + 2$, and $E_{p_i}(L) = -2p_i^2 + 2p_i + 1$. Therefore, the equilibrium action H dominates action M for any $p_i \ge 0.265$ and action L for any $p_i \ge 0$. Given that similar relations also hold for players B and C, a^{*1} is \mathbf{p}_u -dominant with $\mathbf{\bar{p}}_u(a^{*1}) = (0.265, 0.265, 0.265)$. Similar computations show that a^{*2} is \mathbf{p}_u -dominant with $\mathbf{\bar{p}}_u(a^{*2}) = (0.547, 0.547, 0.547)$ while a^{*3} is \mathbf{p}_u -dominant with $\mathbf{\bar{p}}_u(a^{*3}) = (0.771, 0.771, 0.771)$. The \mathbf{p}_u -dominance criterion thus unambiguously selects equilibrium a^{*1} .

2.1 p_u -dominance, risk-dominance and p-dominance

In this section, we investigate the relationships that exist between the \mathbf{p}_u -dominance criterion and the risk-dominance (Harsanyi and Selten, 1988) and \mathbf{p} -dominance (Morris et al., 1995, Kajii and Morris, 1997) criteria. We also explore how \mathbf{p}_u -dominance relates to mixed strategies equilibria.

Lemma 1 In any $2x^2$ coordination game, the \mathbf{p}_u -dominance criterion always selects the risk-dominant equilibrium.

Proof. Consider a generic 2x2 coordination game with $i \in \{A, B\}$ and $A_i = \{H, L\}$

	$a_B = H$	$a_B = L$	
$a_A = H$	a, e	b, f	with $a > c$, $d > b$, $e > f$ and $h > g$.
$a_A = L$	c,g	d,h	

The equilibrium $a^{*1} = (H, H)$ is \mathbf{p}_u -dominant with $\mathbf{\bar{p}}_u(a^{*1}) = \left(\frac{d-b}{a-c+d-b}, \frac{h-g}{e-f+h-g}\right)$, while $a^{*2} = (L, L)$ is \mathbf{p}_u -dominant with $\mathbf{\bar{p}}_u(a^{*2}) = \left(\frac{a-c}{a-c+d-b}, \frac{e-f}{e-f+h-g}\right)$. Therefore, the \mathbf{p}_u -dominant criterion selects a^{*1} if $\frac{d-b}{a-c+d-b} < \frac{a-c}{a-c+d-b}$ and $\frac{h-g}{e-f+h-g} < \frac{e-f}{e-f+h-g}$, i.e., if a-c > d-b and e-f > h-g. But if both conditions are valid then (a-c)(e-f) > (d-b)(h-g), i.e., a^{*1} is the risk-dominant equilibrium because it is the equilibrium characterized by the highest product of the deviation losses. Similarly, if the \mathbf{p}_u -dominant criterion selects a^{*2} , then (d-b)(h-g) > (a-c)(e-f), i.e., a^{*2} is the risk dominant equilibrium.

The proof of Lemma 1 provides an example of a relation that holds more generally in any 2x2 game with multiple equilibria: if an equilibrium $a^{*\psi} \in \{a^{*1}, a^{*2}\}$ is \mathbf{p}_u -dominant with $\bar{\mathbf{p}}_u(a^{*\psi}) = (\bar{p}_1(a^{*\psi}), \bar{p}_2(a^{*\psi}))$ then $\bar{p}_i(a^{*\psi}) = q_j$ with $j \neq i$ where q_j is the probability that defines the mixed strategy equilibrium of the game: $(q_1a_1^{*\psi} + (1-q_1)a_1, q_2a_2^{*\psi} + (1-q_2)a_2)$ with $a_i \neq a_i^{*\psi}$. The intuition is the following: in a mixed strategy equilibrium, agents randomize in such a way as to make the other player indifferent about what to play. This means that if player i attaches probability $\bar{p}_i(a^{*\psi}) = q_j$ to the event of j playing action $a_j^{*\psi}$ (and thus probability $(1-\bar{p}_i(a^{*\psi})) = (1-q_j)$ to the event of j playing a_j given that $k_j - 1 = 1$, and therefore the \mathbf{p}_u -conjecture uniformly randomizes with probability $(1-\bar{p}_i(a^{*\psi}))$ over a single action), then both actions in A_i are best responses for player i. But for any $q'_j > q_j$, action $a_i^{*\psi}$ becomes player i's unique best response. It follows that the equilibrium $a^{*\psi}$ is \mathbf{p}_u -dominant with $\bar{\mathbf{p}}_u(a^{*\psi}) = (\bar{p}_1(a^{*\psi}), \bar{p}_2(a^{*\psi}))$ where $\bar{p}_1(a^{*\psi}) = q_2$ and $\bar{p}_2(a^{*\psi}) = q_1$.

We now move to a comparison of the \mathbf{p}_u -dominance criterion and the \mathbf{p} -dominance criterion. As defined in Morris et al. (1995) for the case with two players and then extended by Kajii and Morris (1997) to the many players case, an equilibrium a^* is \mathbf{p} -dominant with

 $\mathbf{p}(\mathbf{a}^*) = (p_1(a^*), ..., p_n(a^*))$ if, for any agent $i \in N$, action a_i^* is a best response to any probability distribution $\lambda \in \Delta(A_{-i})$ such that $\lambda(a_j^*) \ge p_i(a^*)$ for any $j \neq i$.² In other words, action a_i^* is **p**-dominant if it maximizes player *i*'s expected payoff whenever *i* thinks that each one of the other players will play with probability not smaller than $p_i(a^*)$ his component of the equilibrium profile. The difference with respect to \mathbf{p}_u -dominance is that **p**-dominance does not require the remaining probability $(1 - \lambda(a_j^*))$ to follow any particular distribution over the alternative actions $a_j \neq a_j^*$. As such, the smallest probability vector for which the actions that support a certain equilibrium dominate the alternatives may differ across the two concepts.

Lemma 2 If an equilibrium a^* is **p**-dominant with $\mathbf{\bar{p}}(a^*)$ then it is also \mathbf{p}_u -dominant with $\mathbf{\bar{p}}_u(a^*) \leq \mathbf{\bar{p}}(a^*)$.

Proof. Let a^* be a **p**-dominant equilibrium with $\mathbf{\bar{p}}(a^*) = (\bar{p}_1(a^*), ..., \bar{p}_n(a^*))$ where, for any i and any $j \neq i$, $\bar{p}_i(a^*)$ is the smallest probability for which action a_i^* best responds to any conjecture according to which $\lambda(a_j^*) \geq \bar{p}_i(a^*)$ while the remaining probability $(1 - \lambda(a_j^*))$ can follow any distribution on actions $a_j \neq a_j^*$. The conjecture thus includes the situation in which $\lambda(a_j) = \frac{(1-\lambda(a_j^*))}{k_j-1}$ for any $j \neq i$, $a_j \neq a_j^*$ and $k_j = |A_j|$. Therefore, if a^* is **p**-dominant with $\mathbf{\bar{p}}(a^*)$ then a^* is also \mathbf{p}_u -dominant with $\mathbf{p}_u(a^*) = \mathbf{\bar{p}}(a^*)$. Notice that if $k_j = 2$ for any $j \in N$, then $\mathbf{\bar{p}}_u(a^*) = \mathbf{\bar{p}}(a^*)$ as both **p**-dominance and \mathbf{p}_u -dominance assign probability $(1 - \lambda(a_j^*))$ to the event of j playing action $a_j \neq a_j^*$. Now consider the case in which $k_j > 2$ for some j. Assume there exists at least one action $a_i \in A_i$ that is not (weakly or strictly) dominated by a_i^* .³ Let $\tilde{a}(a_i) = (a_i, \tilde{a}_{-i})$ be the action profile that supports the

³If an undominated action does not exist then a^* is \mathbf{p}_u -dominant as well \mathbf{p} -dominant with $\mathbf{\bar{p}}_u(a^*) = \mathbf{\bar{p}}(a^*) = (0, ..., 0).$

²Tercieux (2006a, 2006b) further extends the concept of **p**-dominance by introducing the notion of **p**-best response set: a set profile $S = (S_1, ..., S_n)$ is a **p**-best response set if for every *i* the set S_i contains an action that best responds to any conjecture that assigns probability of at least p_i to the event that other players select their action from S_{-i} .

(non necessarily unique) outcome for which $u_i(a_i, a_{-i}) - u_i(a_i^*, a_{-i}) > 0$ is maximal. Given that $u_i(a_i, a_{-i}^*) - u_i(a_i^*, a_{-i}^*) \leq 0$ (a^* is a Nash equilibrium), it must be the case that in $\tilde{a}(a_i)$ there is at least one player $j \neq i$ that plays $\tilde{a}_j \neq a_j^*$. The equilibrium a^* is **p**-dominant with $\mathbf{\bar{p}}(a^*) = (\bar{p}_1(a^*), ..., \bar{p}_n(a^*))$ if a_i^* dominates any a_i even under the specific conjecture that assigns probability $(1 - \bar{p}_i(a^*))$ to the event of j playing $\tilde{a}_j \neq a_j^*$. On the other hand, a^* is \mathbf{p}_u -dominant with $\mathbf{\bar{p}}_u(a^*) = (\bar{p}'_1(a^*), ..., \bar{p}'_n(a^*))$ if a_i^* dominates any a_i under the conjecture that assigns probability $\left(\frac{1-\bar{p}'_i(a^*)}{k_j-1}\right)$ to the event of j playing $\tilde{a}_j \neq a_j^*$. Equilibrium a^* cannot be \mathbf{p} -dominant with $\mathbf{p}(a^*) = \mathbf{\bar{p}}_u(a^*)$ given that $(1 - \bar{p}'_i(a^*)) > \left(\frac{1-\bar{p}'_i(a^*)}{k_j-1}\right)$ and the conjecture would thus assign too much probability to the occurrence of the profile $\tilde{a}(a_i)$. It then must be the case that $\bar{p}'_i(a^*) < \bar{p}_i(a^*)$. We can thus conclude that $\mathbf{\bar{p}}_u(a^*) \leq \mathbf{\bar{p}}(a^*)$.

3 Conclusions

This paper introduced the concept of \mathbf{p}_u -dominance and proposed the \mathbf{p}_u -dominance criterion as a tool to refine multiple equilibria in normal form games. The criterion selects the equilibrium whose supporting actions are \mathbf{p}_u -dominant under the less stringent set of beliefs. The intuition for such a choice is that the selected equilibrium is the one that better solves any potential trade-off between the profitability of the equilibrium outcome and the riskiness of playing the supporting action in case some of the opponents deviate.

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