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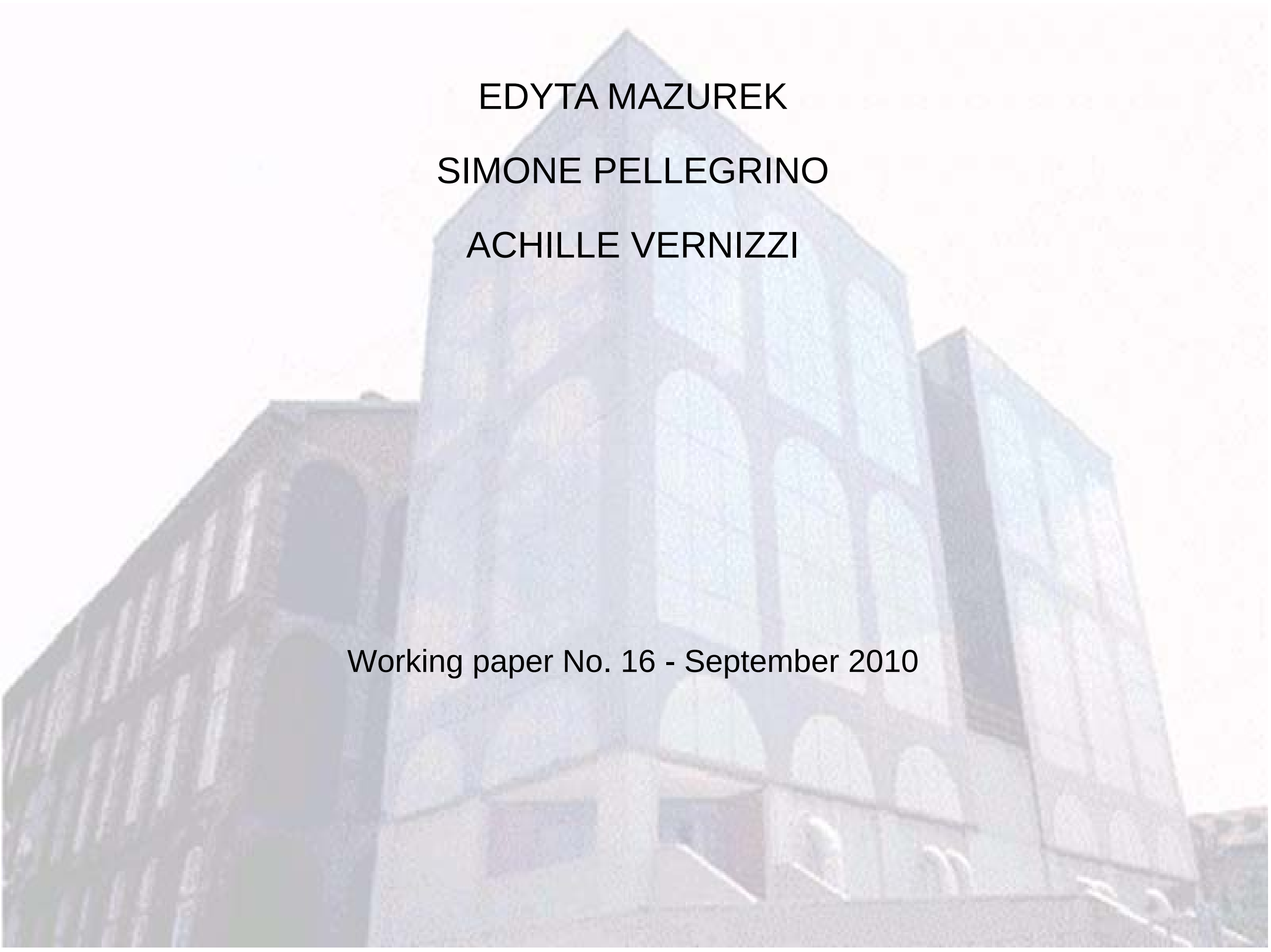

THE DECOMPOSITION OF THE REDISTRIBUTIVE EFFECT AND THE ISSUE OF CLOSE EQUALS IDENTIFICATION

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Abstract

Urban and Lambert (2005, 2008) present an exhaustive summary and an in-depth discussion of the literature contributions about the decomposition of the redistributive effect of a tax (*RE*). The authors discuss the indexes available in the literature for the potential vertical effect (*V*), the loss due to horizontal fairness violations (*H*) and that due to re-rankings (*R*); they also introduce new indexes specifically conceived to take into account problems arising when groups of exact equals are substituted by groups of close equals. Close equals groups are generally obtained by splitting the pre-tax income distribution into contiguous intervals having the same bandwidth, so that the problem of the bandwidth choice arises. van de Van, Creedy and Lambert (2001) suggest choosing the bandwidth that maximizes the potential vertical effect *V*. Even looking for *V* maximization, we discuss a new criterion that yields a compromise between the contrasting needs of minimizing the effects of pre-tax within groups inequalities and the minimization of group average re-rankings. The criterion is then applied to evaluate the components of two decompositions: the former is the one suggested by Urban and Lambert (2005, 2008) as preferable, the latter is suggested by us on the basis of Urban and Lambert's paving discussion. According to our simulation results, when comparing different income tax systems for a same population, the new criterion seems to introduce lower approximation errors than the maximization of *V*.

JEL Classification Numbers: H23; H24.

Keywords: Personal Income Tax, Redistributive Effect, Horizontal Inequity, Re-ranking.

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1. Introduction

Decomposing the income tax redistributive effect across groups of pre-tax equals, into potential vertical, horizontal and re-ranking effects has been intensively studied in recent years.

The original work by Aronson, Johnson and Lambert (1994) considers exact pre-tax equals in portioning the pre-tax income distribution. However, in the real world exact equals are rare: in order to overcome the problem, the suggestion is splitting the pre-tax income distribution into contiguous groups. Income earners contained in a same income interval are considered as “close equals”. Then two problems arise: (*i*) the original decomposition of the redistributive effect has to be adapted to represent situations where exact equals groups are substituted by close equals groups and (*ii*) a proper constant bandwidth has to be chosen to create intervals.

Urban and Lambert (2005, 2008) present an exhaustive summary and an in-depth discussion of the literature contributions about the decomposition of the redistributive effect of a tax system. The authors discuss the indexes available in the literature for the potential vertical effect, the loss due to horizontal fairness violations and that due to re-rankings; they also introduce new indexes specifically conceived to take into account problems arising when groups of exact equals are substituted by groups of close equals.

In this article we suggest a further decomposition, which uses the same vertical effect index suggested by Urban and Lambert, but differs both in the horizontal effect index and in the sources of re-ranking explicitly taken into account.

van de Van, Creedy and Lambert (2001) suggest choosing the bandwidth that maximizes the potential vertical effect: we here propose a new criterion for the choice of the bandwidth. Without neglecting the importance of maximizing the potential vertical effect, this criterion takes into consideration two contrasting needs: (*i*) minimizing the effects of pre-tax within groups inequalities and (*ii*) limiting as much as possible group averages re-ranking between the post-tax and the-pre-tax income distributions. Both the criterion which maximizes the value of the vertical effect and our criterion lead to a specific bandwidth for any tax system. If we want to compare the effects of different tax reforms on a same population of taxpayers, this can be seen as a problem. If we decide to adopt the specific “optimal” bandwidth for each tax system, the decompositions of the redistributive effect may not be fully comparable, as the elements depend on the bandwidth. However, under the hypothesis that the components of the redistributive effect are properly estimated by the bandwidth which is specifically optimal for each tax system, whenever a unique bandwidth is adopted, approximations errors are likely to be introduced. We give empirical evidence of the relative efficiency of the new criterion, testing different tax systems on two different gross income distributions.

The paper is organized as follows: in section 2 we present our decomposition of the redistributive effect; in section 3 we discuss our new criterion for the choice of the bandwidth; in section 4 we describe the simulations which test the efficiency of the new criterion, when different tax systems are analyzed by a unique bandwidth. Section 5 contains concluding remarks.

2. The decomposition of the redistributive effect

Consider a given taxpayers population, with a pre-tax and a post-tax income distribution *X* and *Y*, respectively; let G_X and G_Y are the Gini indexes for the pre-tax and the post-tax income parade. As it is well known, the redistribution effect can be evaluated as

$$RE = G_X - G_Y \,. \tag{1}$$

By making use of Gini index decomposition properties, Aronson, Johnson and Lambert (1994), henceforth AJL, split the actual redistributive effect (1) into the potential vertical effect, the horizontal inequity and the re-ranking effects due to the taxes.

In general, whenever the Gini coefficient is calculated for a population attribute *Z*, and population units can be gathered into groups, G_Z decomposes into the sum of three non-negative components: the between group component G_Z^B, the within group component G_Z^W and the overlapping component. The between group component G_Z^B is obtained by substituting all attribute values within each group by their group average.

The within group component is defined as $G_Z^W = \sum_k a_{k,Z} G_{k,Z}$, where $G_{k,Z}$ is the Gini coefficient for the k-th group, $a_{k,Z}$ is the product of the population share and the attribute share related to the k-th group. The overlapping component can be obtained as a difference by subtracting the sum $\left(G_Z^B + G_Z^W\right)$ to G_Z.

AJL assume that income earners can be gathered into groups of subjects having the same pre-tax income: so, in the pre-tax income distribution, being all incomes within a group equal to one another, the within group and the overlapping components are equal to zero so that $G_X \equiv G_X^B$.

Concerning the post-tax income distribution, AJL assume that group income averages maintain the same ranking they had before taxes, and admit that, in general, both the within group and the overlapping components are no longer necessarily equal to zero, so G_Y assumes the general form $G_Y = G_Y^B + G_Y^W + R^{AJL}$, where, according to Urban and Lambert (2005, 2008) notation, $R^{AJL} = G_Y - \left(G_Y^B + G_Y^W\right)$.

Under these assumptions, AJL can write the redistribution effect as

$$RE = G_X - G_Y = \left(G_X - G_Y^B\right) - G_Y^W - R^{AJL}, \qquad (2)$$

and define $V = \left(G_X - G_Y^B\right)$ as the potential vertical effect, $H = G_Y^W$ as the horizontal (inequity) effect and R^{AJL} as the re-ranking effect: actually H measures how much those who were equal before taxes have become no longer equal after taxes and R^{AJL} coincides with the Atkinson-Plotnick-Kakwani re-ranking index R^{APK}. Given the above enlisted hypotheses, the concentration index for post-tax incomes, aligned according to the pre-tax non-decreasing order, is $D_Y = \left(G_Y^B + G_Y^W\right)$, so that $R^{APK} = G_Y - D_Y$ coincides with R^{AJL}.

However, exact equals groups are in general quite rare, so that the suggestion is to select "close equals groups" (henceforth CEG) by splitting the pre-tax income range into contiguous income intervals having the same bandwidth: close equals are then income earners who have their incomes falling in a same interval. We observe that the bandwidth has to be large enough to gather some incomes and small enough to include nearly equal incomes. Dealing with CEG, G_X^W is now generally different from zero, being a direct function of the adopted bandwidth; the after tax overlapping component, $G_X - \left(G_X^B + G_X^W\right)$, is still equal to zero by construction, as in the pre-tax income distribution groups are contiguous and, then, without intersections.

Moreover, as taxes may alter also the within group ranking and the group averages ranking, the situation may be more complex than that described by equation (2).

When CEG are considered, an immediate generalization of (2) is given by van de Ven, Creedy and Lambert (2001), who suggest the following *RE* decomposition

$$RE = G_X - G_Y = \left(G_X^B - G_Y^B\right) - \left(G_Y^W - G_X^W\right) - R^{AJL} = V^{VCL} - H^{VCL} - R^{AJL}. \tag{3}$$

In (3), the potential vertical effect and the horizontal effect are measured by $V^{VCL} = \left(G_X^B - G_Y^B\right)$ and $H^{VCL} = \left(G_Y^W - G_X^W\right)$, respectively.

In coherence with the principle of CEG, Urban and Lambert (2005, 2008) introduce the idea of smoothing taxes within groups and give a further definition for the vertical and the horizontal effects. If groups contain close equals, their incomes should be taxed by the same tax rate, which can be properly estimated by the group average tax rate. Having applied a same tax rate to all incomes in group *k*, the Gini index for group *k* remains exactly equal to the pre-tax one $G_{k,X}$; however the smoothed within groups Gini coefficient $G_Y^{SW} = \sum_k a_{k,Y} G_{k,X}$ is generally different from $G_X^W = \sum_k a_{k,X} G_{k,X}$, because $a_{k,X} \neq a_{k,Y}$. The authors define the potential vertical effect $V^{AJL} = G_X - \left(G_Y^B + G_Y^{SW}\right)$ and the horizontal effect $H^{AJL} = G_Y^W - G_Y^{SW}$, so that they can decompose *RE* as

$$RE = \left(G_X - G_Y^B - G_Y^{SW}\right) - \left(G_Y^W - G_Y^{SW}\right) - R^{AJL} = V^{AJL} - H^{AJL} - R^{AJL}. \tag{4}$$

We remark the significant difference between H^{VCL} and H^{AJL}: H^{VCL} applies different weights to the pre-tax and post tax Gini coefficient of a same group, being $H^{VCL} = \sum_k \left(a_{k,Y} G_{k,Y} - a_{k,X} G_{k,X}\right)$; conversely, H^{AJL} applies the same weight, being $H^{AJL} = \sum_k a_{k,Y} \left(G_{k,Y} - G_{k,X}\right)$, so that H^{AJL} reflects only (weighed) variations of group Gini indexes, while H^{VCL} depends not only on these variations, but also on differences from pre-tax and post-tax weights. As a consequence, H^{AJL} cannot be positive if all

$G_{k,X} \geq G_{k,Y}$, whereas the sign of H^{VCL} depends also on the combined effect due to $a_{k,Y}$ and $a_{k,X}$. Empirical evidence shows that V^{VCL} dominates V^{AJL} and that V^{VCL} can be still increasing when V^{AJL} has already started decreasing[1].

Whenever re-rankings occur among group averages and within group incomes, R^{AJL} considered in (3) and (4) is only a part of the *APK* re-ranking index[2]: then, in general, as Urban and Lambert observe, R^{APK} can be more generally split into three components

$$R^{APK} = R^{AJL} + R^{B} + R^{W}, \qquad (5)$$

having defined $R^{B} = \left(G_{Y}^{B} - D_{Y}^{B}\right)$ and $R^{W} = \left(G_{Y}^{W} - D_{Y}^{W}\right)$, where D_{Y}^{B} and D_{Y}^{W} are the between and within group concentration coefficients for post-tax income parade, respectively[3].

Urban and Lambert observe that both (3) and (4) do not specify between and within groups re-rankings; for this reason they correct the potential vertical measure V^{AJL} and the horizontal effect H^{AJL}: they add the component R^{B} to V^{AJL} and subtract R^{W} from H^{AJL} obtaining the following decomposition for *RE*

$$RE = \left(G_{X} - D_{Y}^{B} - G_{Y}^{SW}\right) - \left(D_{Y}^{W} - G_{Y}^{SW}\right) - R^{APK} = V^{UL} - H^{UL} - R^{APK}. \qquad (6)$$

In (6) the vertical effect and the horizontal effect are now expressed by $V^{UL} = G_{X} - \left(D_{Y}^{B} + G_{Y}^{SW}\right) = V^{AJL} + R^{B}$ and $H^{UL} = D_{Y}^{W} - G_{Y}^{SW} = H^{AJL} - R^{W}$, respectively; note that V^{UL} can be read as the between groups "full" vertical effect, $\left(G_{X}^{B} - D_{Y}^{B}\right)$, corrected by $G_{Y}^{SW} - G_{X}^{W}$.

The index V^{UL} has the advantage of being part of a decomposition containing all re-ranking components, and not only R^{AJL}; however, as observed in Urban and Lambert (2005, 2008), H^{UL} turns to be negative very soon, even for very small bandwidths. For this reason the authors suggest evaluating a positive version of this index (we label it H^{UL+}): by ordering income series according to their pre-tax order, H^{UL+} is calculated

[1] This is documented by empirical evidence in Urban and Lambert (2008), Mazurek (2009) and Mussini and Zavanella (2009).

[2] We evaluate these violations using both the Bank of Italy survey of households income and wealth and the Wroclaw Municipality tax payers data set.

[3] D_{Y}^{B} is defined as the concentration index when all incomes inside each group are substituted by the group income average and, moreover, groups are ranked according to pre-tax group averages. $D_{Y}^{W} = \sum_{k} a_{k,Y} D_{k,Y}$, where $D_{k,Y}$ is the concentration index for the *k*-th group, when after tax incomes are ranked according to their pre-tax ranking, and $a_{k,Y}$ is the product of the *k*-th group population share and post-tax income share.

considering as positive the areas defined between the concentration curve of post-tax incomes and that of smoothed post-tax incomes; which can cross many times. Unluckily, in general, H^{UL+} does not verify decomposition (6).

We want to remark that the main problem in the decompositions (3) and (4) is not that R^{AJL} is just one of the three components of R^{APK}. The real problem is that, as Vernizzi, Monti and Mussini (2010) point out, when group average re-ranking occurs, R^{AJL} cannot be considered at all as a re-ranking measure. In this case, R^{AJL} simply indicates the presence of income earners that have not permuted their after tax positions with respect to their initial pre-tax positions. The index R^{APK} can be more properly decomposed as the sum of two re-ranking measures: the measure of within groups re-ranking R^{W} and the measure of the across groups re-ranking, that is the re-ranking concerning incomes belonging to different groups, R^{AG}. This last measure may well be decomposed as $R^{AG} = R^{B} + R^{AJL}$, but both R^{B} and R^{AJL} cannot be so clearly interpreted.

On the basis of these considerations, we think that we should surely avoid decompositions of the redistributive effect where the term R^{AJL} appears by itself.

Moreover there are situations where it can be suitable to substitute H^{UL} by H^{AJL} and to exclude R^{W} from the overall re-ranking term. Supposing that within group pre-tax incomes are all different, but differences are so slight which could be due to random measurement errors, even if the post-tax ranking noticeably differs from the pre-tax one either in the numbers of permutations or in their intensities, with an index R^{W} probably relatively quite high, likely it would be improper evaluating this as a case of re-ranking. In our opinion it would be better considered just as a case where the post-tax income inequality becomes greater than the pre-tax one. If we accept this consideration, the re-ranking index could be more properly measured by R^{AG} instead than by R^{APK}. Considering the horizontal effect, H^{AJL} can be a more direct and a bit less biased measure than H^{UL+}, being the former a weighed sum of differences between the post-tax and the pre-tax Gini indexes of each group.

Then we suggest the following decomposition:

$$RE = V^{UL} - H^{AJL} - R^{AG} . \qquad (7)$$

If we now compare expression (7) with expression (6), we observe that in (7) the re-ranking index no longer includes R^W, being $R^{AG} = \left(R^{AJL} + R^B\right)$; moreover the horizontal effect is measured by H^{AJL} as it was in decomposition (4).

In conclusion, we think that if CEG are *close* enough, which is more likely to happen when the groups are small, (7) can well be considered either an alternative or a complement to (6)[4].

Nevertheless, the vertical and the horizontal effect in (6) and (7) depend on bandwidth. Then, the real problem is rather to individuate a convenient bandwidth in order to properly evaluate V's, H's and, in case, R^{APK} components. The next sections try to contribute to the solution of this problem.

3. The "optimal" bandwidth

In the CEG approach there is the problem of determining the bandwidth by which the pre-tax income distribution has to be split into contiguous groups of income. Starting from van de Ven, Creedy and Lambert (2001) it is commonly accepted to choose the bandwidth where V or $\left(V/RE\right)$ is maximum. In their seminal paper, AJL hypothesize that (*i*) exact equals groups could be identified and that (*ii*) no re-ranking occurred for post-tax group averages; we suggest a criterion that chooses a bandwidth that not only draws out as much as possible of the vertical effect, but that also tries to minimize the effects due to within groups inequalities and group averages re-rankings.

If hypotheses (*i*) and (*ii*) are verified, being G_X^W and R^B equal to zero, expression (3) coincides with the original AJL's decomposition (2) and, in particular, $V^{VCL} = G_X^B - G_Y^B$ correctly measures the vertical effect as originally defined by AJL. If we write V^{UL} as

$$V^{UL} = V^{VCL} - \left(G_Y^{SW} - G_X^W\right) + R^B, \qquad (8)$$

we see that V^{UL} can be obtained from to V^{VCL} by subtracting $\left(G_Y^{SW} - G_X^W\right)$ and by adding R^B. The addition of R^B makes V^{VCL}, the between groups vertical effect, to be substituted

[4] Also H^{AJL} becomes negative, when bandwidths are rather large: in Urban and Lambert (2008) for bandwidths larger than 40,000 HRK, in Mussini and Zavanella (2009) for bandwidths larger than 10,000 Euro.

by $\left(G_X^B - D_Y^B\right)$, which is the between groups "full" vertical effect; $\left(G_Y^{SW} - G_X^W\right)$ corrects for the within groups Gini index variation, which depends on pre-tax within groups inequalities.

Observe that the within groups Gini index variation $\left(G_Y^W - G_X^W\right)$, can be expressed as

$$G_Y^W - G_X^W = \left[\sum_k a_{k,Y}\left(G_{k,Y} - G_{k,X}\right)\right] + \left[\sum_k \left(a_{k,Y} - a_{k,X}\right)\cdot G_{k,X}\right], \qquad (9)$$

where $\sum_k a_{k,Y}\left(G_{k,Y} - G_{k,X}\right) = H^{AJL}$ and $\sum_k \left(a_{k,Y} - a_{k,X}\right)\cdot G_{k,X} = \left(G_Y^{SW} - G_X^W\right)$. Given the pre-tax and post-tax system of weights, while the former component, H^{AJL}, depends on groups inequality modifications, the latter, $\left(G_Y^{SW} - G_X^W\right)$, depends on pre-tax within groups inequalities.

Even if V^{UL} is conceived in order to keep into account violations of (*i*) and (*ii*), by the corrections yielded through the components R^B and $\left(G_Y^{SW} - G_X^W\right)$, we think it would be better choosing a bandwidth which needs these corrections as small as possible. We would then suggest looking for a compromise between the minimization of R^B and $\left(G_Y^{SW} - G_X^W\right)$, and the maximisation of V^{UL}. In order to yield this target, we adopt the criterion

$$\min\left[\frac{\max\left\{\left|G_Y^{SW} - G_X^W\right|; R^B\right\}}{V^{UL}}\right]. \qquad (10)$$

Expression (10) considers the absolute value for $\left(G_Y^{SW} - G_X^W\right)$; this is probably an unnecessary precaution. Let's consider

$$V^{VCL} - V^{AJL} = G_Y^{SW} - G_X^W = \sum_k \left(a_{k,Y} - a_{k,X}\right)\cdot G_{k,X} = \sum_k \frac{n_k^2 \mu_k \left(\overline{t} - t_k\right)}{n^2 \mu \left(1 - \overline{t}\right)}\cdot G_{k,X}, \qquad (11)$$

where μ_k is the average of pre-tax incomes, n_k the number of (equivalent) income earners, t_k the average tax rate for group k, μ the overall average of pre-tax incomes, n the total number of (equivalent) income earners and $\overline{t}$ the overall tax rate, respectively. Being income distributions generally skew, groups presenting a positive difference for

$\left(\bar{t} - t_k\right)$ belong to the left tail of the distribution and are more crowded than those which belong to the right tail; as a consequence, $G_Y^{SW} - G_X^W$ is expected to be non-negative, which explains why $V^{VCL} - V^{AJL} \geq 0$ (and, consequently, $H^{VCL} - H^{AJL} \geq 0$)[5].

We observe that R^B and $\left(G_Y^{SW} - G_X^W\right)$ show a contrasting behaviour: R^B appears to be a decreasing function with respect to the bandwidth, while $\left(G_Y^{SW} - G_X^W\right)$ increases with the bandwidth, at least as long as the bandwidth does not become very large; then in the limit both components are zero when the bandwidth coincides with the income range. In any case, when $\left(G_Y^{SW} - G_X^W\right)$ starts decreasing, V^{UL} appears to be already much lower than RE[6].

The behavior of (10) is represented in Figure 1 and 2: for smaller bandwidths, the numerator of (10) coincides with R^B; then, after the minimum, it becomes equal to $\left(G_Y^{SW} - G_X^W\right)$. Empirical evidence shows that (10) presents an asymmetric *U*-shaped form, with its minimum in a neighbourhood of the bandwidth where R^B and $G_Y^{SW} - G_X^W$ lines cross[7].

Both the criterion which maximizes the value of the vertical effect and the criterion (10) lead to a specific bandwidth for any tax system. However if we want to analyze the effects of different tax reforms on a same population of taxpayers, we should adopt a unique bandwidth in order to have fully comparable estimates for the components of the redistributive effect.

In the next section, by using different microsimulation models, we test the efficiency of criterion (10) with respect to the maximization of either V^{UL} or V^{AJL}, when one bandwidth has to be adopted. We will consider the components entering decomposition (6) and (7), which depend on the bandwidth.

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5 Urban and Lambert (2005, 2008), Vernizzi and Pellegrino (2008), Mazurek (2009), Mussini and Zavanella (2009).

6 Mussini and Zavanella (2009) and Mazurek (2009) show that $G_Y^{SW} - G_X^W$ starts to decrease when the bandwidth is really large: larger than 50.000/100.000 Pl zl in Mazurek's empirical analysis, and larger than 50.000 € in Mussini and Zavanella's one.

7 We again refer to Mussini and Zavanella (2009) and Mazurek (2009) for an extensive empirical analysis of the behaviour of (10) as bandwidth enlarges. Even if the authors consider a ratio that is different from (10), the author's ratio and the ratio at expression (10) converge as V^{UL} and V^{AJL} converge.

4. The efficiency of the new criterion when comparing different tax systems

When the effects of tax reforms are to be analyzed, one needs to apply a same bandwidth in order to be able to compare indexes that are functions of the bandwidth. This need however contradicts the adoption of a criterion which would identify a proper bandwidth for each tax system. In fact, if one assumes that the "true" indexes are those calculated in correspondence of the optimal bandwidth, indexes calculated in correspondence of bandwidths different than the optimal one are only approximations of the "true" ones. Then a criterion should be evaluated also under the aspect of its efficiency in containing the approximation errors which arise when a unique bandwidth has to be adopted for different tax-systems.

In this section we will compare the behavior of criterion (10) with that of the criterion that maximizes the vertical effect, when different tax systems are considered. We will take into consideration the components V^{UL}, H^{UL+}, H^{AJL} and R^{AG} that enter decompositions (6) and (7). The maximization criterion is applied both to V^{AJL} and V^{UL}; the maximization of V^{VCL} is not taken into consideration: as it appears in Urban and Lambert (2005, 2008), Mazurek (2009) and in Mussini and Zavanella (2009), the maximum for V^{VCL} is reached for quite large bandwidths, so that income groups can hardly be considered as CEG. In what follows, we indicate the minimization of the ratio given at (10) by *mr*, the maximization of V^{AJL} by *MAJL* and the maximization of V^{UL} by *MUL*.

The comparisons are performed by simulations based on gross income distributions concerning both Italy and the Municipality of Wrocław (Poland). The gross income distribution for Italy is obtained through a micro-simulation model based on the 2006 Bank of Italy Survey on household income and wealth (Pellegrino *et al*, 2010), while that for the Municipality of Wrocław is a 2001 data set kindly made available by Lower-Silesian tax offices. We decided to perform the simulations on the bases of these two data-sets, due to the different characteristics they present: the Polish pre-tax income distribution presents greater inequality indexes, a stronger right skewness and a heavier right tail than the Italian one.[8]

A set of different tax systems (see appendix for details) are applied to gross incomes: ten for the Italian data set and sixteen for the Polish one; the Italian set and the Polish one are considered separately.

For each criterion c $\left(c = MUL, MAJL, mr\right)$, we can obtain a set of N bandwidths, $\left[b_1^c, b_2^c, ..., b_i^c, ..., b_N^c\right]$ each being "optimal" for one of the N tax systems. We indicate by $\theta\left(b_i^c \mid s\right)$ the value assumed by the RE component θ in tax system s at bandwidth b_i^c; in particular $\theta\left(b_s^c \mid s\right)$ indicates the value assumed by θ at the bandwidth b_s^c that criterion c gives as "optimal" for tax-system s.

In the first experiment we assume that, given the criterion c $\left(c = MUL, MAJL, mr\right)$, $\theta\left(b_s^c \mid s\right)$, evaluated at the "optimal" bandwidth b_s^c, is the "true" value of θ, and that the other $\left(N-1\right)$ $\theta\left(b_i^c \mid s\right)$ are just estimates of $\theta\left(b_s^c \mid s\right)$. The efficiency of criterion c can then be evaluated by the root mean square error of the estimates with respect to the "true" value:

$$RMS^c\left\{\theta \mid s\right\} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[\theta\left(b_i^c \mid s\right) - \theta\left(b_s^c \mid s\right)\right]^2}, \quad (12)$$

$RMS^c\left\{\theta \mid s\right\}$ can be calculated for each tax system s.

The relative efficiency of mr criterion with respect to MUL and $MAJL$ is then defined, respectively, as

$$e_{mr}^{MUL} = \left[RMS^{MUL}\left\{\theta \mid s\right\} / RMS^{mr}\left\{\theta \mid s\right\}\right] \quad (13)$$

$$e_{mr}^{MAJL} = \left[RMS^{MAJL}\left\{\theta \mid s\right\} / RMS^{mr}\left\{\theta \mid s\right\}\right] \quad (14)$$

According to expression (13) and (14), mr is more efficient than MUL and $MAJL$, whenever e_{mr}^{MUL} and e_{mr}^{MAJL} are greater than 1.

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[8] For pre-tax income distribution of the Bank of Italy data-set, the summary statistics are: G_X = 0.438, Coefficient of Variation = 1.18, Skewness = 9.85, Kurtosis = 197.24. Even if we do not take into account the extreme pre-tax incomes, which are appear in the right tail of the Polish distribution, for the Municipality of Wrocław the correspondent statistics are: G_X = 0.483, Coefficient of Variation = 2.79, Skewness = 97.54, Kurtosis = 13,482.55.

Table 3 and 4 summarize the behaviour of (13) and (14) for the Italian and Polish distributions, respectively.

In the second experiment we assume that "true" values are just those obtained at the bandwidth that maximizes either V^{AJL} or V^{UL}. In this experiment criterion (10) is then considered just an instrument to obtain estimates for the true values and it is deliberately disadvantaged with respect to the maximizing criterion, to better evaluate its potentialities.

Then, for what concerns criterions *MAJL* and *MUL*, they continue in being evaluated by $RMS^{c}\{\theta \mid s\}$ defined at (12), conversely, in what concerns *mr*, we now use

$$RMS^{mr}\{\theta \mid s, c\} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[\theta\left(b_i^{mr} \mid s\right) - \theta\left(b_s^{c} \mid s\right)\right]^2} \tag{15}$$

being $c = MUL, MAJL$.

The efficiency of *mr* criterion, with respect to *MUL* and *MAJL*, is now given by the ratios

$$e_{mr|MUL}^{MUL} = \left[RMS^{MUL}\{\theta \mid s\} / RMS^{mr}\{\theta \mid s, MUL\}\right] \tag{16}$$

$$e_{mr|MAJL}^{MAJL} = \left[RMS^{MAJL}\{\theta \mid s\} / RMS^{mr}\{\theta \mid s, MAJL\}\right]. \tag{17}$$

Again, according to (16) and (17), *mr* is more efficient than *MUL* and *MAJL*, whenever $e_{mr|MUL}^{MUL}$ and $e_{mr|MAJL}^{MAJL}$, respectively, are greater than 1.

Table 5 summarizes the behaviour of (16) and (17) for the Italian simulated tax systems and Table 6 summarizes the results obtained by the simulations performed using the Polish data base.

As we can see from Tables 3 and 4, in the first experiment where the three criterions are compared on a same basis, *mr*, the criterion given by the minimization of ratio (10), results everywhere to be more efficient than the criterion of maximization both of V^{AJL} and V^{UL}, being (13) and (14) always greater than 1 and in most cases (always in Table 3) even greater than 2.

In the second experiment, where *mr* is just an instrument to get estimates for the true values and then it "competes with a handicap" with respect to *MUL* and *MAJL*, as we can see, from Tables 5 and 6, *mr* no longer dominates *MUL* and *MAJL* everywhere.

However, even in the second experiment, in general, *mr* remains preferable to *MUL* and *MAJL*. The geometric averages of (16) and (17) are everywhere greater than 1. Moreover, we not only observe that the cases where (16) and (17) are greater than 2 are many more than those where they are lower than 1, but we can also observe that the maximum gains attained by *mr* are much greater than the maximum losses: (16) and (17) can reach quite high values and they are never lower than 0.71.[9]

We can conclude the criterion suggested in this paper can be considered an improvement in choosing bandwidth that has to be robust with respect to changes in post-tax income distributions, as it is the case when comparing of a sequence of tax reforms, concerning a same population of tax payers.

5. Concluding Remarks

On the basis of Urban and Lambert's (2005, 2008) paving discussion, in this article we have introduced a further decomposition of the redistributive effect *RE* and we have suggested a new criterion for the bandwidth choice. We have then evaluated the relative efficiency of the new criterion in minimizing the mean square of the approximation errors, which arise when comparing different tax systems for a same population by a unique bandwidth, instead of the one optimal the maximization of the potential vertical effect has been taken as benchmark.

Our decomposition, presented in section 2, uses the same index suggested by Urban and Lambert to measure the potential vertical effect; however, (*i*) it measures the horizontal effect by the weighed sum of within groups Gini index variations (the H^{AJL} index, according to Urban and Lambert's notation) and (*ii*) the re-ranking effect considers only across groups re-rankings, which is measured by adding the between groups re-ranking index to the after tax overlapping index (the R^{AJL} index, according to Urban and Lambert's notation), or by subtracting the within groups re-ranking index R^W from the Aronson-Plotnick-Kakwani re-ranking index.

[9] The relative efficiency of the new criterion can depend on the different extent of regularity that the curves of $\left(\max\left\{\left|G_Y^{SW} - G_X^W\right|; R^B\right\} / V^{UL}\right)$, V^{AJL} and V^{UL} present (Figures 1-6). Mazurek (2009) analyzes the curves V^{AJL} and V^{UL} for the Polish data set, together with that of a ratio which behaves very close to the one in criterion (10): she measures how the ratio is actually much more regular around its minimum, than both *V's* are around their maxima.

We think that when close equals groups inequalities are particularly slight, this decomposition can be used in alternative or in addition to Urban and Lambert's.

The second contribution of this paper concerns the choice of the so called optimal bandwidth. Starting from van de Ven, Creedy and Lambert (2001) it is commonly accepted to choose the bandwidth where the potential vertical effect is maximized. With the aim of preserving as much as possible the original framework conceived by Aronson, Johnson and Lambert (1994), we have introduced a criterion that, without neglecting the importance of maximizing the potential vertical effect, becomes a compromise of two different and contrasting needs, if taken together: (*i*) the minimization of the within groups Gini index variation that depends on pre-tax within groups inequalities, (*ii*) the minimization of group average re-ranking.

In section 4 we consider the problem that arises when the effects of different tax systems on a same population are to be analyzed and, consequently, one needs to apply a same bandwidth, in order to have indexes that can be comparable. This need however contradicts the adoption of a criterion which would identify a proper bandwidth for each tax system. If one assumes that the "true" indexes are those calculated in correspondence of the optimal bandwidth, the indexes calculated in correspondence of bandwidths different than the optimal one are only approximations of the "true" ones.

By microsimulations performed using both the Bank of Italy Survey on household income and wealth and the data set of tax payers in the Municipality of Wrocław (Poland), we have shown that the (root) mean square errors of the approximations are always lower, and often much lower, when the indexes are calculated in correspondence of bandwidths identified by the new criterion, than by the criterion which maximizes the potential vertical effect. Moreover, the bandwidths identified by the new criterion give better approximation in average even when we assume that the true indexes are those calculated in correspondence of bandwidths identified by the maximizing criterion.

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Table 1: Summary of index definitions

CEG	CEG (close equals groups) are constituted by subjects belonging to a same pre-tax income bracket; income brackets are created by splitting the pre-tax non decreasing incomes parade into contiguous intervals characterized by a same income bandwidth. Groups contain the same subjects both before and after taxes, whatever ordering criterion is adopted. Before taxes no overlapping exists by construction; taxation may result in group overlapping.
G_X	Gini index for pre-tax income parade.
G_X^B	between groups Gini index for pre-tax income parade: it is defined as the Gini index when all incomes inside each group are substituted by the group income average.
G_X^W	within groups Gini index for pre-tax income parade: $G_X^W = \sum_k a_{k,X} G_{k,X}$, where $G_{k,X}$ is the Gini index for the *k*-th group and $a_{k,X}$ is the product of the *k*-th group population share and pre-tax income share.
G_Y	Gini index for post-tax income parade.
G_Y^B	it is analogous to G_X^B for the post-tax income parade.
G_Y^W	within groups Gini index for post-tax income parade: $G_Y^W = \sum_k a_{k,Y} G_{k,Y}$, where $G_{k,Y}$ is the post-tax Gini index for the *k*-th group and $a_{k,Y}$ is the product of the *k*-th group population share and post-tax income share.
D_Y	concentration index for post-tax income parade when ordered according to the pre-tax order.
D_Y^B	between groups concentration index for post-tax income parade: it is defined as the concentration index when all incomes inside each group are substituted by the group income average, moreover groups are ordered according to pre-tax group averages.
D_Y^W	within groups concentration index for post-tax income parade: $D_Y^W = \sum_k a_{k,Y} D_{k,Y}$; $D_{k,Y}$ is the concentration index for the *k*-th group, when the *k*-th group incomes are ordered according to the pretax within group order, and $a_{k,Y}$ is the product of the *k*-th group population share and post-tax income share.
G_Y^{SW}	within groups Gini index for post-tax smoothed income parade. The smoothed income parade is obtained by applying the group average tax rate to all incomes which belong to the same group. $G_Y^{SW} = \sum_k a_{k,Y} G_{k,X}$, as the Gini index for the *k*-th group remains unchanged, when all group incomes are taxed by a same tax rate.

Table 2: Summary of equations and components

$R^{APK} = R^{AJL} + R^{B} + R^{W}$ $R^{B} = \left(G_{Y}^{B} - D_{Y}^{B}\right)$ $R^{W} = \left(G_{Y}^{W} - D_{Y}^{W}\right)$ $\lim_{b\to 0} R^{AJL} = 0 \quad \lim_{b\to 0} R^{B} = R^{APK} \quad \lim_{b\to 0} R^{W} = 0$ $\lim_{b\to MAX} R^{AJL} = 0 \quad \lim_{b\to MAX} R^{B} = 0 \quad \lim_{b\to MAX} R^{W} = R^{APK}$
$RE = V^{VCL} - H^{VCL} - R^{AJL}$ $V^{VCL} = G_{X}^{B} - G_{Y}^{B}$ $H^{VCL} = G_{Y}^{W} - G_{X}^{W}$ $R^{AJL} = G_{Y}^{t} = G_{Y} - G_{Y}^{B} - G_{Y}^{W}$ $\lim_{b\to 0} V^{VCL} = RE \quad \lim_{b\to 0} H^{VCL} = 0 \quad \lim_{b\to 0} R^{AJL} = 0$ $\lim_{b\to MAX} V^{VCL} = 0 \quad \lim_{b\to MAX} H^{VCL} = -RE \quad \lim_{b\to MAX} R^{AJL} = 0$
$RE = V^{AJL} - H^{AJL} - R^{AJL}$ $V^{AJL} = G_{X} - G_{Y}^{B} - G_{Y}^{SW} =$ $= V^{VCL} - \left(G_{Y}^{SW} - G_{X}^{W}\right)$ $H^{AJL} = G_{Y}^{W} - G_{Y}^{SW}$ $\lim_{b\to 0} V^{AJL} = RE \quad \lim_{b\to 0} H^{AJL} = 0 \quad \lim_{b\to 0} R^{AJL} = 0$ $\lim_{b\to MAX} V^{AJL} = 0 \quad \lim_{b\to MAX} H^{AJL} = -RE \quad \lim_{b\to MAX} R^{AJL} = 0$
$RE = V^{UL} - H^{UL} - R^{APK}$ $V^{UL} = G_{X} - D_{Y}^{B} - G_{Y}^{SW} =$ $= V^{AJL} + R^{B}$ $= V^{VCL} - \left(G_{Y}^{SW} - G_{X}^{W}\right) + R^{B}$ $H^{UL} = D_{Y}^{W} - G_{Y}^{SW} = H^{AJL} - R^{W}$ $\lim_{b\to 0} V^{UL} = G_{X} - D_{Y} \quad \lim_{b\to 0} H^{UL} = 0$ $\lim_{b\to MAX} V^{UL} = 0 \quad \lim_{b\to MAX} H^{UL} = D_{Y} - G_{X}$

Figure 1: Ratio Italy

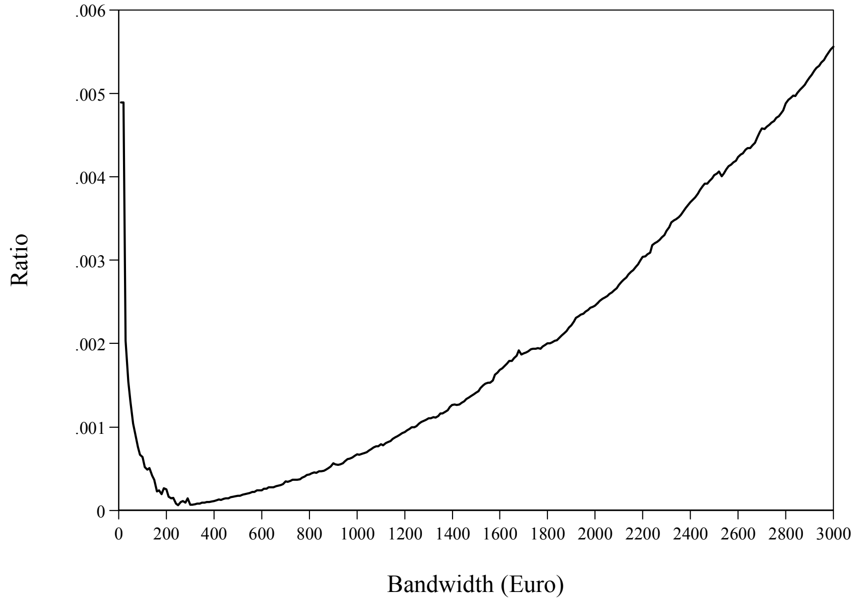

Figure 2: Ratio Poland

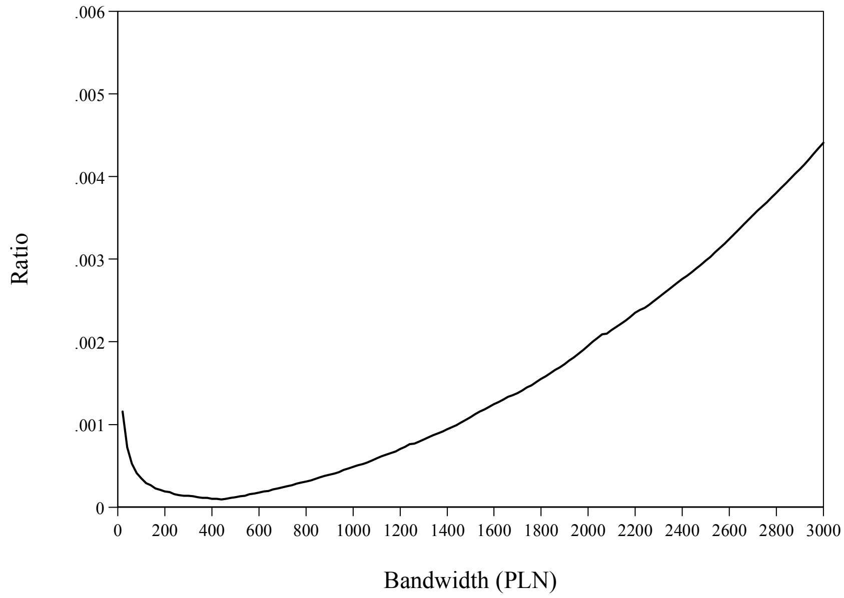

Figure 3: V^{VCL}, V^{AJL} and V^{UL} for 2006 Italian taxpayers (% of *RE*)

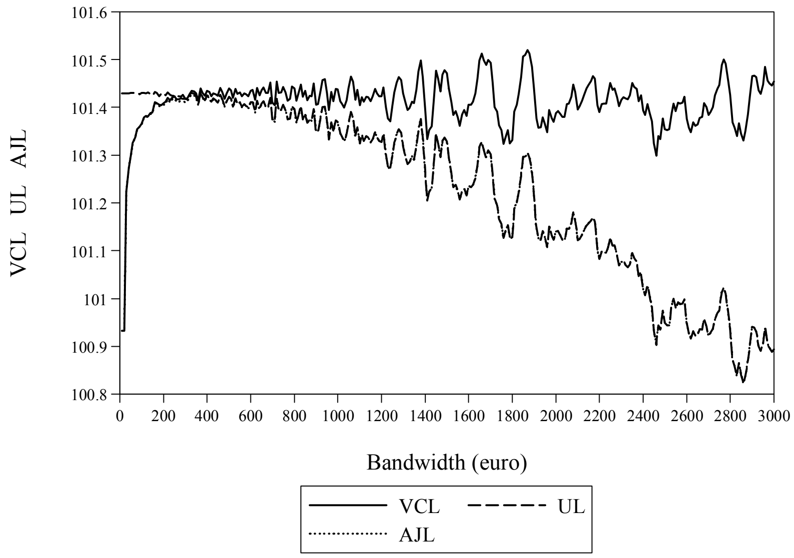

Figure 4: V^{VCL}, V^{AJL} and V^{UL} for 2006 Italian taxpayers (focus) (% of *RE*)

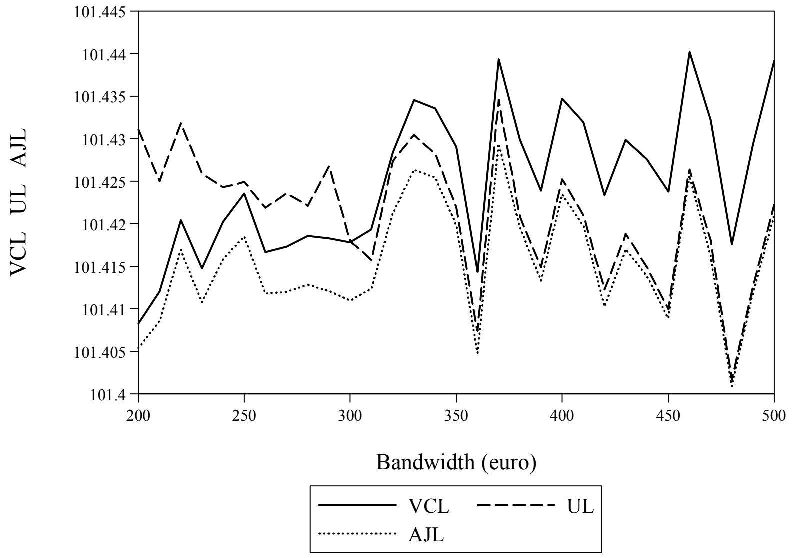

Figure 5: V^{VCL}, V^{AJL} and V^{UL} for 2006 Polish taxpayers (% of *RE*)

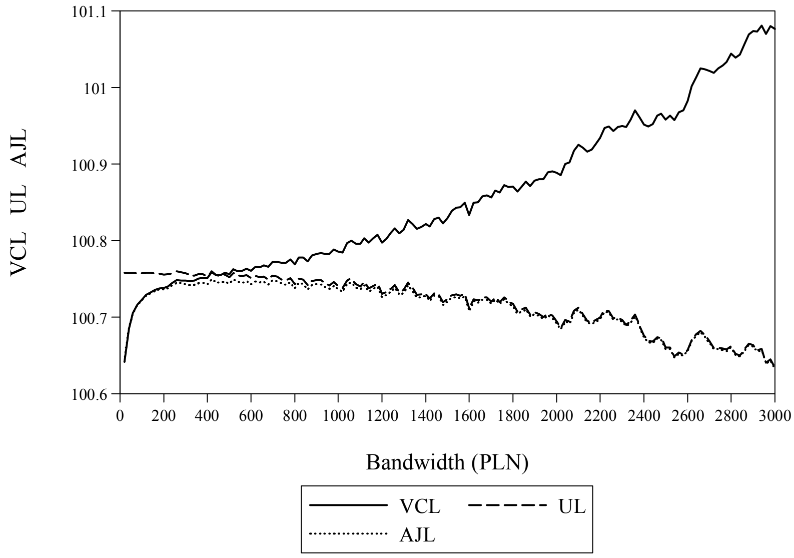

Figure 6: V^{VCL}, V^{AJL} and V^{UL} for 2006 Polish taxpayers (focus) (% of *RE*)

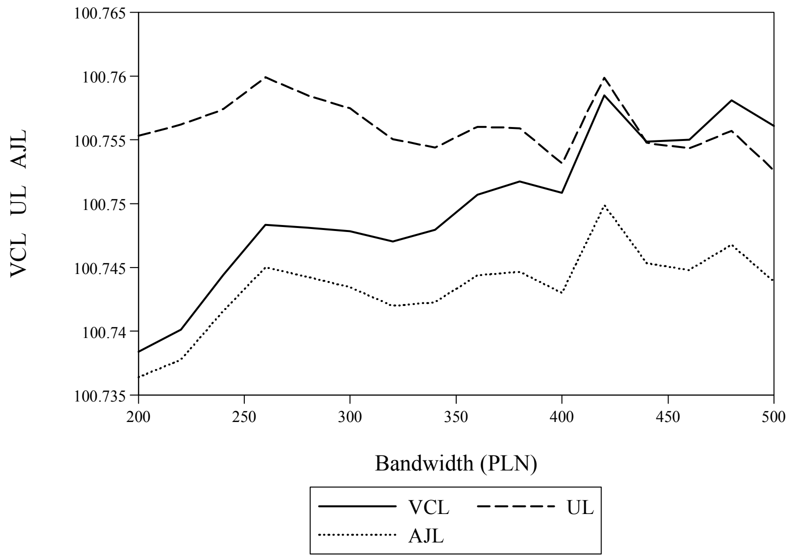

Table 3: Efficiency of *mr* criterion in the Italian simulated tax systems: each criterion provides its own "true" indexes

	V^{UL}		H^{UL+}		H^{AJL}		R^{AG}	
	e_{mr}^{MUL}	e_{mr}^{MAJL}	e_{mr}^{MUL}	e_{mr}^{MAJL}	e_{mr}^{MUL}	e_{mr}^{MAJL}	e_{mr}^{MUL}	e_{mr}^{MAJL}
Max	42.69	42.47	40.31	39.62	8.49	9.82	12.17	9.95
min	2.79	2.74	4.53	4.06	3.34	2.40	3.80	3.18
geometric mean	19.63	19.55	24.26	23.41	5.71	4.65	6.65	5.07
n. of cases>2	10	10	10	10	10	10	10	10
n. of cases 1÷2	0	0	0	0	0	0	0	0
n. of cases 0.5÷1	0	0	0	0	0	0	0	0
n. of cases ≤ 0.5	0	0	0	0	0	0	0	0

Table 4: Efficiency of *mr* criterion in the Polish simulated tax systems: each criterion provides its own "true" indexes

	V^{UL}		H^{UL+}		H^{AJL}		R^{AG}	
	e_{mr}^{MUL}	e_{mr}^{MAJL}	e_{mr}^{MUL}	e_{mr}^{MAJL}	e_{mr}^{MUL}	e_{mr}^{MAJL}	e_{mr}^{MUL}	e_{mr}^{MAJL}
Max	90.07	89.56	83.43	81.89	54.81	55.88	8.98	5.92
min	1.96	2.66	4.14	2.59	2.31	1.50	1.66	1.03
geometric mean	29.94	31.08	28.82	25.76	5.73	4.19	4.02	2.38
n. of cases>2	15	16	16	16	16	11	15	12
n. of cases 1÷2	1	0	0	0	0	5	1	4
n. of cases 0.5÷1	0	0	0	0	0	0	0	0
n. of cases ≤ 0.5	0	0	0	0	0	0	0	0

Table 5: Efficiency of *mr* criterion in the Italian simulated tax systems: "true" indexes are derived by *MUL* and *MAJL* criterions

	V^{UL}		H^{UL+}		H^{AJL}		R^{AG}	
	$e_{mr\|MUL}^{MUL}$	$e_{mr\|MAJL}^{MAJL}$	$e_{mr\|MUL}^{MUL}$	$e_{mr\|MAJL}^{MAJL}$	$e_{mr\|MUL}^{MUL}$	$e_{mr\|MAJL}^{MAJL}$	$e_{mr\|MUL}^{MUL}$	$e_{mr\|MAJL}^{MAJL}$
Max	30.73	38.79	21.64	39.13	2.35	8.78	2.91	5.67
min	0.73	0.71	0.96	0.91	1.07	0.87	1.09	0.89
geometric mean	12.84	15.24	7.68	12.41	1.45	2.74	1.80	3.05
n. of cases>2	8	8	8	8	1	7	2	7
n. of cases 1÷2	1	1	1	1	9	1	8	1
n. of cases 0.5÷1	1	1	1	1	0	2	0	2
n. of cases ≤ 0.5	0	0	0	0	0	0	0	0

Table 6: Efficiency of *mr* criterion in the Polish simulated tax systems: "true" indexes are derived by *MUL* and *MAJL* criterions

	V^{UL}		H^{UL+}		H^{AJL}		R^{AG}	
	$e^{MUL}_{mr\|MUL}$	$e^{MAJL}_{mr\|MAJL}$	$e^{MUL}_{mr\|MUL}$	$e^{MAJL}_{mr\|MAJL}$	$e^{MUL}_{mr\|MUL}$	$e^{MAJL}_{mr\|MAJL}$	$e^{MUL}_{mr\|MUL}$	$e^{MAJL}_{mr\|MAJL}$
Max	43.32	89.56	33.15	74.27	14.98	93.37	3.20	3.13
min	0.93	1.02	1.20	0.84	0.93	0.76	0.83	0.77
geometric mean	14.30	21.32	11.06	16.06	1.86	2.66	1.41	1.57
n. of cases >2	12	12	13	12	5	7	3	4
n. of cases 1÷2	2	4	3	2	8	5	10	8
n. of cases 0.5÷1	2	0	0	2	3	4	3	4
n. of cases $\leq$ 0.5	0	0	0	0	0	0	0	0

APPENDIX

Two different approaches have been adopted in order to obtain different unequal treatment of equals and re-ranking. The ten tax structures applied to the Italian case consider rate schedules and the actual tax allowances and tax credits for items of expenditure as well as income related tax credits. On the contrary, the simulations performed on the Polish data set are based on four basic tax systems applied to real gross incomes, each disturbed by adding a random term, so that sixteen different tax structures have been considered.

The tax structures hypothesized for Italy are the following:

SYSTEM 1: Very progressive system with 21 brackets and tax rates ranging from 3 to 85 per cent. Only tax allowances and tax credits for items of expenditure are allowed.

SYSTEM 2. A 20 per cent flat tax rate. Only tax allowances and tax credits for items of expenditure are allowed.

SYSTEM 3. System with three brackets and three tax rates (10, 30 and 50 per cent). Only tax allowances and tax credits for items of expenditure are allowed.

SYSTEM 4. A 30 per cent tax rate. In addition to tax allowances and tax credits for items of expenditure, an income related tax credit of 1,000 euro is added. It is linearly decreasing with income and become zero above 100 thousand euro.

SYSTEM 5. System equals to system 3 with an income related tax credit as in system 4.

SYSTEM 6. System equals to system 2 with an income related tax credit of 500 euro. It is linearly decreasing with income and become zero above 50 thousand euro.

SYSTEM 7. Progressive system with 9 brackets and tax rates ranging from 10 to 75 per cent. Only tax allowances and tax credits for items of expenditure are allowed.

SYSTEM 8. System equals to system 3 with an income related tax credit as in system 6.

SYSTEM 9. System equals to system 7 with an income related tax credit as in system 6.

SYSTEM 10. A 70 per cent tax rate. Only tax allowances and tax credits for items of expenditure are allowed.

The basic tax structures hypothesized for Poland are the following ones:

BASIC SYSTEM 1. One 15 per cent tax rate is applied to all incomes, all tax payers benefit 556.02 PLN tax credit.

BASIC SYSTEM 2. System with three income brackets: i) 19% from 0 to 44,490 PLN, ii) 30% from 44,490 to 85,528 PLN, all tax payers benefit 586.85 PLN tax credit.

BASIC SYSTEM 3. System with two income brackets: i) 18% from 0 to 85,528 PLN, ii) 32% over 85,528 PLN; all income earners benefit 556.02 PLN tax credit.

BASIC SYSTEM 4. System with four income brackets: i) 10% from 0 to 20.000 PLN, ii) 20% from 20,000 to 40,000 PLN, iii) 30% from 40,000 to 90,000 PLN, iv) 40% over 90,000 PLN; all incomes benefit 500.00 PLN tax credit.

Tax T_i that results after the application of a tax system, is then modified by a random factor Z_i, so that net income becomes $\left(y_i - T_i\right) + Z_i \cdot T_i$. Z_i is drawn from the uniform distributions (*a*) $Z \sim U\left(-0.2 \div 0.2\right)$, (*b*) $Z \sim U\left(0 \div 0.4\right)$, and from the normal distributions (*c*) $Z \sim N(0\,;0.0133)$, (*d*) $Z \sim N(0\,;0.12)$, so that each basic system generates four sub-systems. When the normal distribution is applied, the random factor Z_i is considered in absolute value; the programme did not allow incomes to become either negative or greater that $2y_i$.

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